AN EQUIVARIANT BRAUER GROUP AND ACTIONS OF GROUPS ON C^*-ALGEBRAS

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Abstract. Suppose that (G, T) is a second countable locally compact transformation group given by a homomorphism $\ell : G \to \mathrm{Homeo}(T)$, and that A is a separable continuous-trace C^*-algebra with spectrum T. An action $\alpha : G \to \mathrm{Aut}(A)$ is said to cover ℓ if the induced action of G on T coincides with the original one. We prove that the set $\mathrm{Br}_G(T)$ of Morita equivalence classes of such systems forms a group with multiplication given by the balanced tensor product: $[A, \alpha][B, \beta] = [A \otimes_{C_0(T)} B, \alpha \otimes \beta]$, and we refer to $\mathrm{Br}_G(T)$ as the Equivariant Brauer Group.

We give a detailed analysis of the structure of $\mathrm{Br}_G(T)$ in terms of the Moore cohomology of the group G and the integral cohomology of the space T. Using this, we can characterize the stable continuous-trace C^*-algebras with spectrum T which admit actions covering ℓ. In particular, we prove that if $G = \mathbb{R}$, then every stable continuous-trace C^*-algebra admits an (essentially unique) action covering ℓ, thereby substantially improving results of Raeburn and Rosenberg.

1. Introduction

In 1963, Dixmier and Douady associated to each continuous-trace C^*-algebra A with spectrum T a class $\delta(A)$ in the cohomology group $H^3(T; \mathbb{Z})$, which determines A up to a natural equivalence relation [11, 9]. Over the past 15 years, it has become clear that this relation is precisely the C^*-algebraic version of Morita equivalence developed by Rieffel; this observation appears, for example, in [12, 2], and a modern treatment of the theory is discussed in [35, §3]. It was also realized in the mid–1970's that the results of [11, 9] effectively establish an isomorphism between a Brauer group $\mathrm{Br}(T)$ and $H^3(T; \mathbb{Z})$: the elements of $\mathrm{Br}(T)$ are Morita equivalence classes $[A]$ of continuous-trace algebras A with spectrum T, the multiplication is given by the balanced C^*-algebraic tensor product $[A][B] = [A \otimes_{C(T)} B]$, the identity is $[C_0(T)]$, and the inverse of $[A]$ is represented by the conjugate algebra $\overline{A}$. This point of view was discussed by Taylor [42] and Green [12], although neither published details.

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Much of the current interest in operator algebras focuses on C^*-dynamical systems, in which a locally compact group acts on a C^*-algebra, and it is natural to try to extend the Dixmier-Douady theory to accommodate group actions. Thus one starts with an action of a locally compact group G on a locally compact space T, and considers systems (A,α) in which A is a continuous-trace C^*-algebra with spectrum T and α is an action of G on A which induces the given action of G on T. There is a notion of Morita equivalence for dynamical systems due to Combes [7] and Curto–Muhly–Williams [8], which is easily modified to respect the identifications of spectra with T, and the elements of our equivariant Brauer group $\mathrm{Br}_G(T)$ are the Morita equivalence classes $[A,\alpha]$ of the systems (A,α). The group operation is given by $[A,\alpha][B,\beta] = [A\otimes_{C(T)} B, \alpha\otimes_{C(T)}\beta]$, the identity is $[C_0(T),\tau]$, where $\tau_s(f)(x) = f(s^{-1}\cdot x)$, and the inverse of $[A,\alpha]$ is $[\overline{A},\overline{\alpha}]$, where $\overline{\alpha}(\overline{a}) := \overline{\alpha(a)}$. Even though the key ideas are all in [9], it is not completely routine that $\mathrm{Br}_G(T)$ is a group, and we have to work quite hard to establish that $(A\otimes_{C(T)}\overline{A}, \alpha\otimes_{C(T)}\overline{\alpha})$ is Morita equivalent to $(C_0(T),\tau)$.

Similar Brauer groups have been constructed by Parker for $G = \mathbb{Z}/2\mathbb{Z}$ [25], and by Kumjian in the context of r-discrete groupoids [17]. The results of the preceding paragraph are contained in those of [17] when the group is discrete. However, Kumjian then generalizes the Dixmier–Douady Theorem by identifying his Brauer group with the equivariant cohomology group $H^2(T,G;\mathcal{S})$ of Grothendieck [13]. (If G is trivial, $H^2(T,\mathcal{S})$ is naturally isomorphic to $H^3(T;\mathbb{Z})$, and the original Dixmier–Douady construction proceeds through $H^2(T,\mathcal{S})$.) Grothendieck developed powerful techniques for computing his equivariant cohomology, and there is in particular a spectral sequence $\{E_r^{p,q}\}$ with $E_2^{p,q} = H^p\big(G, H^q(T,\mathcal{S})\big)$ (the group cohomology of G with coefficients in the sheaf cohomology of T) which converges to $H^{p+q}(T,G;\mathcal{S})$. In view of Kumjian's result, this gives a filtration of the equivariant Brauer group $\mathrm{Br}_G(T)$ for discrete G.

For locally compact groups, the appropriate version of group cohomology is the Borel cochain theory developed by Moore [20, 21]. (Computing the 2-cocycle for the extension $0\to\mathbb{Z}\to\mathbb{R}\to\mathbb{T}\to 1$ shows that continuous cochains will not suffice.) The coefficient modules in Moore's theory must be Polish groups, and there are not enough injective objects in this category to allow the direct application of homological algebra, so any suitable generalization of Grothendieck's theory will, at best, be hard to work with. However, we are only interested here in the Brauer group, and the filtration involves only the low-dimensional cohomology groups $H^p\big(G, H^q(T,\mathcal{S})\big)$ for $p = 0,1,2,3$ and $q = 0,1,2$. Each of the coefficient groups $H^0(T,\mathcal{S}) = C(T,\mathbb{T})$, $H^1(T,\mathcal{S})\cong H^2(T;\mathbb{Z})$ and $H^2(T,\mathcal{S})\cong H^3(T;\mathbb{Z})$ admits a C^*-algebraic interpretation: $H^2(T,\mathcal{S})$ is itself the Brauer group of continuous-trace algebras with spectrum T, $H^1(T,\mathcal{S})$ is the group $\mathrm{Aut}_{C_0(T)} A/\operatorname{Inn} A$ of outer $C(T)$-automorphisms of a stable continuous-trace algebra A with spectrum T [28], and $C(T,\mathbb{T})$ is the unitary group $UZM(A)$ of the center of the multiplier algebra $M(A)$ of such an algebra A. Further, the Moore cohomology groups $H^2\big(G, C(T,\mathbb{T})\big)$ and $H^3\big(G, C(T,\mathbb{T})\big)$ arise naturally in the analysis of group actions on a continuous-trace algebra A with spectrum T: H^2 contains the obstructions to implementing an action $\alpha : G\to \mathrm{Inn}(A)$ by a unitary group $u : G\to UM(A)$ [31, §0], and H^3 the obstructions to implementing a homomorphism $\beta : G\to \mathrm{Aut}(A)/\operatorname{Inn}(A)$ by a twisted action (see Lemma 4.6 below). The remarkable point of the present paper

is that, using these interpretations, we have been able to define all the groups and homomorphisms necessary to completely describe the filtration of $\mathrm{Br}_G(T)$ predicted by the isomorphism $\mathrm{Br}_G(T) \cong H^2(T, G, \mathcal{S})$ of the discrete case. Thus we will prove:

Theorem (cf. Theorem 5.1 below). *Let (G, T) be a second countable locally compact transformation group with $H^2(T; \mathbb{Z})$ countable. Then there is a composition series $\{0\} \leq B_1 \leq B_2 \leq B_3 = \mathrm{Br}_G(T)$ of the equivariant Brauer group in which B_3/B_2 is isomorphic to a subgroup of $H^3(T; \mathbb{Z})$, B_2/B_1 to a subgroup of $H^1\big(G, H^2(T; \mathbb{Z})\big)$, and B_1 to a quotient of $H^2\big(G, C(T, \mathbb{T})\big)$. Further, we can precisely identify the subgroups and quotients in terms of homomorphisms between groups of the form $H^p\big(G, H^q(T; \mathbb{Z})\big)$.*

The sting of this theorem lies in, first, the specific nature of the isomorphisms, and, second, in the last sentence, where the homomorphisms are all naturally defined using the C^*-algebraic interpretations of Moore and Čech cohomology. The isomorphism F of B_3/B_2 into $H^3(T; \mathbb{Z})$ takes $[A, \alpha]$ to $\delta(A)$, so its kernel B_2 is the set of classes of the form $[C_0(T, \mathcal{K}), \alpha]$. For the isomorphism of B_2 into $H^1\big(G, H^2(T; \mathbb{Z})\big)$, we use the exact sequence

$$0 \longrightarrow \mathrm{Inn}\, C(T, \mathcal{K}) \longrightarrow \mathrm{Aut}_{C(T)}\, C(T, \mathcal{K}) \xrightarrow{\ \zeta\ } H^2(T; \mathbb{Z}) \longrightarrow 0$$

of [28], and send $(C_0(T, \mathcal{K}), \alpha) \in B_2$ to the cocycle $s \mapsto \zeta(\alpha_s)$ in $Z^1\big(G, H^2(T; \mathbb{Z})\big)$. Thus B_1 consists of the systems $(C_0(T, \mathcal{K}), \alpha)$ in which $\alpha : G \to \mathrm{Inn}\, C_0(T, \mathcal{K})$, and the last isomorphism takes such an action α to its Mackey obstruction—the class in $H^2\big(G, H^2(T; \mathbb{Z})\big)$ which vanishes precisely when α is implemented by a unitary group $u : G \to UM(A)$.

To illustrate the second point, we describe our identification of the range of the first homomorphism $F : \mathrm{Br}_G(T) \to H^3(T; \mathbb{Z})$. We first restrict attention to the group $H^3(T; \mathbb{Z})^G$ of classes fixed under the canonical action of G, and define a homomorphism $d_2 : H^3(T; \mathbb{Z})^G \to H^2\big(G, H^2(T; \mathbb{Z})\big)$. We then define another homomorphism d_3 from the kernel of d_2 to a quotient of $H^3\big(G, C(T, \mathbb{T})\big)$, and prove that the image of F is the kernel of d_3. To see why this is powerful, note that a stable algebra A with spectrum T carries an action of G covering the given action on T if and only if $\delta(A) \in \mathrm{Im}\, F$. When $G = \mathbb{R}$, $H^3(T; \mathbb{Z})^{\mathbb{R}} = H^3(T; \mathbb{Z})$, and results from [31] show that $H^3\big(\mathbb{R}, C(T, \mathbb{T})\big) = H^2\big(\mathbb{R}, H^2(T; \mathbb{Z})\big) = 0$; we deduce that F maps onto $H^3(T; \mathbb{Z})$, and hence that *every* action of $\mathbb{R}$ on T lifts to an action of $\mathbb{R}$ on *every* stable continuous-trace algebra A with spectrum T (see Corollary 6.1 below). This is a substantial generalization of results proved in [31, §4]—and even they required considerable machinery.

We should stress that, even when there is no group action and T is compact, our Brauer group $\mathrm{Br}(T)$ is not the usual Brauer group of the commutative ring $C(T)$, which is isomorphic to the torsion subgroup of $H^3(T; \mathbb{Z})$ rather than $H^3(T; \mathbb{Z})$ [14]. Although the two groups have different objects, $\mathrm{Br}(T)$ is isomorphic to the bigger Brauer group $\widetilde{B}\big(C(T)\big)$ of Taylor [43, 32], which is a purely algebraic invariant designed to accommodate non-torsion cohomology classes. Presumably there is also an equivariant version of $\widetilde{B}(R)$ for which theorems similar to ours are true—indeed, the results in [32, 17] suggest that $\widetilde{B}_G(R)$ might then be isomorphic to an equivariant étale cohomology group.

Our work is organized as follows. In Section 2 we outline some of the basic definitions of the internal and external tensor products of imprimitivity bimodules which are fundamental to our approach. In Section 3 we discuss the Morita equivalence of systems, define our Brauer group, and prove that it is indeed a group. We then devote Section 4 to identifying the range of our Forgetful Homomorphism $F : \mathrm{Br}_G(T) \to \mathrm{Br}(T) \cong H^3(T;\mathbb{Z})$, which is probably the most important part of our main theorem. In Section 5, we give a precise statement of our theorem, and finish off its proof. In the last section, we discuss the application to actions of $\mathbb{R}$, and consider some special cases in which we can say more about $\mathrm{Br}_G(T)$.

We will adopt the following conventions. When we consider a *C^*-algebra A with spectrum T* we are considering a pair (A, ϕ) where $\phi : \hat{A} \to T$ is a fixed homeomorphism. While we have opted to be less pedantic and drop the ϕ, it is necessary to keep its existence in mind. Thus, as in [35, §2], we will work almost exclusively with complete imprimitivity bimodules which preserve the spectrum: if A and B are C^*-algebras with spectrum T, then $\mathcal{X}$ is an $A -_T B$-imprimitivity bimodule if $\mathcal{X}$ is an imprimitivity bimodule in the usual sense and the Rieffel homeomorphism $h_{\mathcal{X}} : T \to T$ is the identity. It is convenient to keep in mind that, if A and B have continuous trace, then if follows from Proposition 1.11 and the preceding remarks in [30] that $h_{\mathcal{X}} = \mathrm{id}$ if and only if the left and right actions of $C_0(T)$ on $\mathcal{X}$, induced by the actions of A and B, respectively, coincide: i.e., $\phi \cdot x = x \cdot \phi$ for all $\phi \in C_0(T)$ and $x \in \mathcal{X}$. (See [35, §2] for further details.) We will also make full use of dual imprimitivity bimodules as defined in [38, Definition 6.17]. Recall that if $\mathcal{X}$ is an $A -_T B$-imprimitivity bimodule, the dual $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ is the set $\{\tilde{x} : x \in \mathcal{X}\}$, made into a $B -_T A$-imprimitivity bimodule as follows:

$$b \cdot \tilde{x} = (x \cdot b^*)^\sim \qquad \tilde{x} \cdot a = (a^* \cdot x)^\sim$$
$${}_B\langle \tilde{x}, \tilde{y} \rangle = \langle x, y \rangle_B \qquad \langle \tilde{x}, \tilde{y} \rangle_A = {}_A\langle x, y \rangle,$$

for $x, y \in \mathcal{X}$, $a \in A$, and $b \in B$.

We will use the notation $H^n(T;\mathbb{Z})$ for the ordinary integral cohomology groups, and $H^n(T, \mathcal{S})$ for the sheaf cohomology groups with coefficients in the sheaf of germs of continuous circle-valued functions on T. We will make frequent use of the canonical isomorphism of $H^2(T, \mathcal{S})$ and $H^3(T;\mathbb{Z})$; in particular, we will view the Dixmier-Douady class $\delta(A)$ of a continuous-trace C^*-algebra with spectrum T as belonging to whichever of these groups is more convenient for the matter at hand. It will also be essential to use Moore's Borel cochain version of group cohomology as presented in [20]: when G is a locally compact group, and A is a Polish G-module, $H^n(G, A)$ will denote the corresponding Moore group.

The construction of our equivariant Brauer group was originally intended to be part of the first author's Ph.D. thesis; in particular, Theorem 3.6 is basically due to him. Much of this work was carried out while the first three authors were at the University of New South Wales. It was finished while the third author was visiting the University of Colorado, and he thanks his colleagues there for their warm hospitality. This research was supported by the Australian Research Council.

2. Tensor products of imprimitivity bimodules

Let A, B, C, and D be C^*-algebras. Suppose that $\mathcal{X}$ is a $A-B$-imprimitivity bimodule and that $\mathcal{Y}$ is a $B-C$-imprimitivity bimodule. Then the algebraic tensor product $\mathcal{X} \odot \mathcal{Y}$ is a $A-C$-bimodule and carries A- and C-valued inner products defined, respectively, by

$$\langle\!\langle x \otimes y, x' \otimes y' \rangle\!\rangle_C = \big\langle \langle x', x\rangle_B y, y' \big\rangle_C \tag{2.1}$$

$$_A\langle\!\langle x \otimes y, x' \otimes y' \rangle\!\rangle = {}_A\big\langle x, x'\, {}_B\langle y', y\rangle \big\rangle. \tag{2.2}$$

It is straightforward to verify that $\mathcal{X} \odot \mathcal{Y}$ is a (pre-) $A-C$-imprimitivity bimodule, and we shall write $\mathcal{X} \otimes_B \mathcal{Y}$ for the completion with respect to the common semi-norm induced by the inner products (see [39, §3]). Suppose that in addition A, B, and C have spectrum (identified with) T, and that $\mathcal{X}$ is a $A -_T B$-imprimitivity bimodule and $\mathcal{Y}$ is a $B -_T C$-imprimitivity bimodule. Then it is shown in [30, Lemma 1.3], that $\mathcal{X} \otimes_B \mathcal{Y}$ is a $A -_T C$-imprimitivity bimodule. Although $\mathcal{X} \otimes_B \mathcal{Y}$ is not a Banach space tensor product in the usual sense, it does follow from [38, Proposition 2.9] that

$$\|x \otimes y\| \le \|x\|\,\|y\|. \tag{2.3}$$

The construction above is an example of an *internal tensor product* of Hilbert modules as described in [16, §1.2]. We will also need the *external tensor product*. Specifically, if $\mathcal{X}$ is a $A-C$-imprimitivity bimodule and $\mathcal{Y}$ is a $B-D$-imprimitivity bimodule, then the formulas

$$_{A\otimes B}\langle\!\langle x \otimes y, x' \otimes y' \rangle\!\rangle = {}_A\langle x, x'\rangle \otimes {}_B\langle y, y'\rangle \tag{2.4}$$

$$\langle\!\langle x \otimes y, x' \otimes y' \rangle\!\rangle_{C\otimes D} = \langle x, x'\rangle_C \otimes \langle y, y'\rangle_D \tag{2.5}$$

define, respectively, $A \odot B$- and $C \odot D$-valued sesqui-linear forms on $\mathcal{X} \odot \mathcal{Y}$. It follows from [16, 1.2.4] that these forms are inner products for any C^*-norms on $A \odot B$ and $C \odot D$, and that in particular $\mathcal{X} \odot \mathcal{Y}$ can be completed to a $A \otimes B - C \otimes D$-imprimitivity bimodule (recall that '$\otimes$' denotes that minimal tensor product)[1]. In order to more clearly distinguish which tensor product of imprimitivity bimodules we're using, we shall write $\mathcal{X} \mathbin{\widehat{\otimes}} \mathcal{Y}$ for the completion of $\mathcal{X} \odot \mathcal{Y}$ with respect to the operations in (2.4) and (2.5).

Now suppose that A, B, C, and D have Hausdorff spectrum T and that $\mathcal{X}$ is a $A -_T C$-imprimitivity bimodule and $\mathcal{Y}$ a $B -_T D$-imprimitivity bimodule. In particular, by the Dauns-Hofmann Theorem, $C_0(T)$ sits in the center of the multiplier algebras of all these algebras so that $\mathcal{X}$ and $\mathcal{Y}$ are $C_0(T)$-bimodules. Therefore we can form the balanced tensor products $A \otimes_{C_0(T)} B$ and $C \otimes_{C_0(T)} D$. Each of these algebras has spectrum T (cf., e.g., [33, Lemma 1.1]). Recall that $A \otimes_{C_0(T)} B$ is the quotient of $A \otimes B$ by the closed ideal I_T spanned by $\{\, \phi \cdot a \otimes b - a \otimes \phi \cdot b :$

[1] This is observed in [3, §13.5], and in [5, Proposition 2.9] where it is also observed that the same holds for the maximal tensor product. In general, if ν is a C^*-norm on $C \odot D$, then $A \odot B$ acts as adjointable bounded operators with respect to the right Hilbert $C \otimes_\alpha D$-module structure on $\mathcal{X} \odot \mathcal{Y}$. This provides $A \odot B$ with a C^*-norm ν^* for which the completion of $\mathcal{X} \odot \mathcal{Y}$ is a $A \otimes_{\nu^*} B - C \otimes_\nu D$-imprimitivity bimodule. Since all our algebras will be continuous-trace C^*-algebras, and hence nuclear, the result from [16] will suffice.

$a \in A$, $b \in B$, and, $\phi \in C_0(T)\,\}$. Similarly, $C \otimes_{C_0(T)} D$ is the quotient of $C \otimes D$ by an ideal J_T.

Lemma 2.1. *Suppose that A, B, C, and D are C^*-algebras with Hausdorff spectrum T, and that $\mathcal{X}$ and $\mathcal{Y}$ are, respectively, $A\,\text{–}_T\,B$- and $C\,\text{–}_T\,D$-imprimitivity bimodules. Then the correspondence of* [39, §3] *between ideals in $C \otimes D$ and ideals in $A \otimes B$, induced by $\mathcal{X} \mathbin{\widehat{\otimes}} \mathcal{Y}$, maps I_T to J_T. In particular, the corresponding quotient $\mathcal{X} \mathbin{\widehat{\otimes}}_{C_0(T)} \mathcal{Y}$ of $\mathcal{X} \mathbin{\widehat{\otimes}} \mathcal{Y}$ is a $A \otimes_{C_0(T)} B\,\text{–}_T\,C \otimes_{C_0(T)} D$-imprimitivity bimodule.*

Proof. Let $K(I_T)$ be the ideal of $C \otimes D$ corresponding to I_T via the Rieffel correspondence K. Since $I_T \cdot (\mathcal{X} \mathbin{\widehat{\otimes}} \mathcal{Y})$ is spanned by vectors of the form

$$(\phi \cdot a \otimes b - a \otimes \phi \cdot b) \cdot (x \otimes y) = \big((\phi \cdot a) \cdot x \otimes b \cdot y\big) - \big(a \cdot x \otimes (\phi \cdot b) \cdot y\big),$$

where $a \in A$, $b \in B$, $\phi \in C_0(T)$, $x \in \mathcal{X}$, and $y \in \mathcal{Y}$, it follows that

$$K(I_T) \subseteq \overline{\operatorname{span}}\{\, \langle\!\langle v, u \rangle\!\rangle_{C \otimes D} : v \in V_0, u \in \mathcal{X} \odot \mathcal{Y} \,\},$$

where $V_0 = \operatorname{span}\{\, \phi \cdot x \otimes y - x \otimes \phi \cdot y : \phi \in C_0(T), x \in \mathcal{X}, y \in \mathcal{Y} \,\}$. Consequently, $K(I_T) \subseteq J_T$. By symmetry, we have $J_T \subseteq K(I_T)$, and therefore $K(I_T) = J_T$, which proves the first assertion.

The second assertion will follow from the first and the discussion preceding the lemma once we show that the left and right $C_0(T)$-actions on the quotient module $(\mathcal{X} \mathbin{\widehat{\otimes}} \mathcal{Y})/I_T \cdot (\mathcal{X} \mathbin{\widehat{\otimes}} \mathcal{Y})$ coincide. But

$$\phi \cdot [x \otimes y] = [\phi \cdot x \otimes y] = [x \cdot \phi \otimes y] = [x \otimes y] \cdot \phi.$$

(The first equality holds because $\phi \cdot (a \otimes b) = (\phi \cdot a \otimes b) = (a \otimes \phi \cdot b)$ in $A \otimes_{C_0(T)} B$, and the second because the module $\mathcal{X}$ is T-balanced by assumption. The third is similar to the first.) $\square$

The next result is implicit in [9]. Our approach here views the Dixmier-Douady class $\delta(A)$ of a continuous-trace C^*-algebra A as the obstruction to the existence of a global Morita equivalence of A with $C_0(T)$ as described in [35, §3].

Proposition 2.2. *Suppose that A and B are continuous-trace C^*-algebras with spectrum T. Then $\delta(A \otimes_{C_0(T)} B) = \delta(A) + \delta(B)$.*

Proof. Since A has continuous trace, it follows from [35, Lemmas 6.1 and 6.2] that there are compact sets $F_i \subseteq T$ whose interiors form a cover $\mathfrak{A} = \{\, \operatorname{int} F_i : i \in I \,\}$ of T such that:

(1) for each $i \in I$ there are $A^{F_i}\,\text{–}_{F_i}\,C(F_i)$-imprimitivity bimodules $\mathcal{X}_i$, and
(2) for each $i, j \in I$, there are imprimitivity bimodule isomorphisms $g_{ij} : \mathcal{X}_j^{F_{ij}} \to \mathcal{X}_i^{F_{ij}}$.

Then the class $\delta(A)$ in $H^3(T, \mathbb{Z})$ is determined by the cocycle $\nu = \{\, \nu_{ijk} \,\}$ in $\check{H}^2(\mathfrak{A}, \mathcal{S})$ defined by

$$g_{ij}^{F_{ijk}}\big(g_{jk}^{F_{ijk}}(x)\big) = \nu_{ijk} \cdot g_{ik}^{F_{ijk}}(x).$$

By taking refinements, we may assume that we have similar data for B consisting of bimodules $\{\, \mathcal{Y}_i \,\}$, isomorphisms $\{\, h_{ij} \,\}$, and a cocycle $\mu = \{\, \mu_{ijk} \,\}$ all defined with respect to the same cover $\mathfrak{A}$.

The result follows from verifying that $(A \otimes_{C_0(T)} B)^{F_i} \cong A^{F_i} \otimes_{C(F_i)} B^{F_i}$, that $(\mathcal{X}_i \mathbin{\widehat{\otimes}}_{C(F_i)} \mathcal{Y}_i)^{F_{ij}} \cong \mathcal{X}_i^{F_{ij}} \mathbin{\widehat{\otimes}}_{C(F_{ij})} \mathcal{Y}_i^{F_{ij}}$, and that $k_{ij} = g_{ij} \otimes h_{ij}$ defines an isomorphism of $\mathcal{X}_j^{F_{ij}} \mathbin{\widehat{\otimes}}_{C(F_{ij})} \mathcal{Y}_j^{F_{ij}}$ onto $\mathcal{X}_i^{F_{ij}} \mathbin{\widehat{\otimes}}_{C(F_{ij})} \mathcal{Y}_i^{F_{ij}}$. Then $k_{ij}^{F_{ijk}} \circ k_{jk}^{F_{ijk}} = \nu_{ijk}\mu_{ijk} \cdot h_{ik}^{F_{ijk}}$. $\square$

3. The Brauer Group

For the remainder of this article, (G,T) will be a second countable locally compact transformation group. We define $\mathfrak{Br}_G(T)$ to be the class of pairs (A,α) where A is a separable continuous-trace C^*-algebra with spectrum T and $\alpha : G \to \operatorname{Aut}(A)$ is a strongly continuous action inducing the given action τ on $C_0(T)$. That is, $\alpha_s(\phi \cdot a) = \tau_s(\phi) \cdot \alpha_s(a)$ for $a \in A$ and $\phi \in C_0(T)$, where $\tau_s(\phi)(t) = \phi(s^{-1} \cdot t)$.

We say that two elements (A,α) and (B,β) of $\mathfrak{Br}_G(T)$ are equivalent, written $(A,\alpha) \sim (B,\beta)$, if they are Morita equivalent over T in the sense of Combes [7] (see also [35, §4]): this means that there is an A–$_T\,B$-imprimitivity bimodule $\mathcal{X}$ and an action u of G on $\mathcal{X}$ by linear transformations, which is strongly continuous (i.e., $s \mapsto u_s(x)$ is norm-continuous for all x) and satisfies

$$\alpha_s\bigl({}_A\langle x,y\rangle\bigr) = {}_A\bigl\langle u_s(x), u_s(y)\bigr\rangle \quad\text{and}\quad \beta_s\bigl(\langle x,y\rangle_B\bigr) = \bigl\langle u_s(x), u_s(y)\bigr\rangle_B.$$

We claim that $\sim$ is an equivalence relation. It is certainly reflexive: take $(\mathcal{X},u) = (A,\alpha)$. Symmetry is immediate from the existence of dual imprimitivity bimodules: one only has to define $\tilde{u}_s$ by $\tilde{u}_s(\tilde{x}) = (u_s(x))^{\sim}$. Transitivity requires more work. Suppose that $(A,\alpha) \sim (B,\beta)$ via $(\mathcal{X},u)$ and that $(B,\beta) \sim (C,\gamma)$ via $(\mathcal{Y},v)$. Then we have

$$\begin{aligned}
\Bigl\| u_s \otimes v_s\Bigl(\sum_i x_i \otimes y_i\Bigr)\Bigr\|^2 &= \Bigl\|\sum\nolimits_i u_s(x_i) \otimes v_s(y_i)\Bigr\|^2 \\
&= \Bigl\|\sum\nolimits_{i,j} \bigl\langle \langle u_s(x_j), u_s(x_i)\rangle_B v_s(y_i), v_s(y_j)\bigr\rangle_C\Bigr\| \\
&= \Bigl\|\sum\nolimits_{i,j} \bigl\langle \beta_s\bigl(\langle x_j, x_i\rangle_B\bigr) \cdot v_s(y_i), v_s(y_j)\bigr\rangle_C\Bigr\| \qquad (3.1)\\
&= \Bigl\|\sum\nolimits_{i,j} \bigl\langle v_s\bigl(\langle x_j, x_i\rangle_B \cdot y_i\bigr), v_s(y_j)\bigr\rangle_C\Bigr\|
\end{aligned}$$

which, because v_s is isometric, is

$$\begin{aligned}
&= \Bigl\|\sum\nolimits_{i,j} \bigl\langle \langle x_j, x_i\rangle_B \cdot y_i, y_j\bigr\rangle_C\Bigr\| \\
&= \Bigl\|\sum\nolimits_i x_i \otimes y_i\Bigr\|^2.
\end{aligned}$$

Therefore $w_s = u_s \otimes v_s$ defines an action of G on $\mathcal{X} \otimes_B \mathcal{Y}$, which is strongly continuous in view of (2.3), and $(\mathcal{X} \otimes_B \mathcal{Y}, w)$ provides the required equivalence between (A,α) and (C,γ). We will write $\operatorname{Br}_G(T)$ for the set $\mathfrak{Br}_G(T)/{\sim}$ of equivalence classes[2].

[2]Notice that $\operatorname{Br}_G(T)$ is actually a set. To see this, fix an infinite dimensional separable Hilbert space $\mathcal{H}$. Then the separable C^*-subalgebras of $B(\mathcal{H})$ form a set as do all possible actions of G on each subalgebra, and therefore as do the collection of all possible actions on separable subalgebras of $B(\mathcal{H})$. This set contains a representative for each equivalence class. (Separability could be replaced by any fixed cardinality.)

It will be helpful to keep in mind that the above equivalence relation can be reformulated as follows. Recall that two actions $\alpha : G \to \mathrm{Aut}(A)$ and $\beta : G \to \mathrm{Aut}(B)$ are *outer conjugate* if there is an isomorphism $\Phi : A \to B$ so that β is exterior equivalent to $\Phi \circ \beta \circ \Phi^{-1}$. We say that α and β are *stably outer conjugate* if $\alpha \otimes i$ and $\beta \otimes i$ are outer conjugate as actions on $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$, respectively. If A and B have spectrum T, then we say that α is outer conjugate over T if Φ can be taken to $C_0(T)$-linear.

Lemma 3.1. *Suppose that $(A, \alpha), (B, \beta) \in \mathfrak{Br}_G(T)$. Then $(A, \alpha) \sim (B, \beta)$ if and only if α is stably outer conjugate to β over T. If A and B are both stable, then $(A, \alpha) \sim (B, \beta)$ if and only if α is outer conjugate to β over T.*

Proof. The first statement follows from the second since $(A, \alpha) \sim (A \otimes \mathcal{K}, \alpha \otimes i)$ and $(B, \beta) \sim (B \otimes \mathcal{K}, \beta \otimes i)$. But, if A and B are stable, and $(A, \alpha) \sim (B, \beta)$, then the proposition in §9 of [7] implies that α and β are outer conjugate. The argument in the last paragraph of the proof of Lemma 2.3 in [37] shows that the isomorphism of A onto B can be taken to be $C_0(T)$-linear. Finally, if α and β are outer conjugate over T, then the other half of the same proposition implies that (A, α) is Morita equivalent to (B, β), and again, it is straightforward to see that we can take the Morita equivalence over T. $\square$

Let (A, α) and (B, β) be elements of $\mathfrak{Br}_G(T)$. Notice that $\alpha_s \otimes \beta_s(\phi \cdot a \otimes b - a \otimes \phi \cdot b) = (\phi \circ \tau_s^{-1}) \cdot \alpha_s(a) \otimes \beta_s(b) - \alpha_s(a) \otimes (\phi \circ \tau_s^{-1}) \cdot \beta_s(b)$. Thus, $\alpha_s \otimes \beta_s$ maps the closed ideal I_T of $A \otimes B$ spanned by $\{ \phi \cdot a \otimes b - a \otimes \phi \cdot b \}$ to itself, and defines an automorphism $\alpha_s \otimes_{C_0(T)} \beta_s$ of $A \otimes_{C_0(T)} B = A \otimes B / I_T$. It is easy to check that $\alpha_s \otimes_{C_0(T)} \beta_s$ induces the given action on T, so that $(A \otimes_{C_0(T)} B, \alpha \otimes_{C_0(T)} \beta) \in \mathfrak{Br}_G(T)$. For notational convenience, we will usually write $\alpha \otimes \beta$ rather than $\alpha \otimes_{C_0(T)} \beta$.

Lemma 3.2. *Suppose that $(A, \alpha) \sim (C, \gamma)$ via $(\mathcal{X}, u)$ and that $(B, \beta) \sim (D, \delta)$ via $(\mathcal{Y}, v)$. Then $(A \otimes_{C_0(T)} B, \alpha \otimes \beta)$ is equivalent to $(C \otimes_{C_0(T)} D, \gamma \otimes \delta)$ in $\mathfrak{Br}_G(T)$.*

Proof. As pointed out in Section 2, $\mathcal{X} \mathbin{\widehat{\otimes}}_{C_0(T)} \mathcal{Y}$ is an $A \otimes_{C_0(T)} B\text{–}_T\, C \otimes_{C_0(T)} D$-imprimitivity bimodule. The argument that $w_s(x \mathbin{\widehat{\otimes}} y) = u_s(x) \mathbin{\widehat{\otimes}} v_s(y)$ gives a well-defined strongly continuous action of G on $\mathcal{X} \mathbin{\widehat{\otimes}}_{C_0(T)} \mathcal{Y}$ is similar, but more straightforward, than (3.1) above. Then $(\mathcal{X} \mathbin{\widehat{\otimes}}_{C_0(T)} \mathcal{Y}, w)$ is the required $(A \otimes_{C_0(T)} B, \alpha \otimes \beta)\text{–}_T\, (C \otimes_{C_0(T)} D, \gamma \otimes \delta)$-imprimitivity bimodule. $\square$

Proposition 3.3. *The binary operation*

$$[A, \alpha][B, \beta] = [A \otimes_{C_0(T)} B, \alpha \otimes \beta] \tag{3.2}$$

is well-defined on $\mathrm{Br}_G(T)$, and with respect to this operation, $\mathrm{Br}_G(T)$ is a commutative semi-group with identity equal to the class of $(C_0(T), \tau)$.

Proof. The operation (3.2) is well-defined by virtue of Lemma 3.2. Since an equivariant $C_0(T)$-isomorphism of A onto B certainly gives a Morita equivalence over T, associativity and commutativity follow from the observations that $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$ and $a \otimes b \mapsto b \otimes a$ define equivariant $C_0(T)$-isomorphisms of $(A \otimes_{C_0(T)} B) \otimes_{C_0(T)} C$ onto $A \otimes_{C_0(T)} (B \otimes_{C_0(T)} C)$ and $A \otimes_{C_0(T)} B$ onto $B \otimes_{C_0(T)} A$, respectively. Similarly, $(C_0(T), \tau)$ is an identity because the isomorphism $a \otimes f \mapsto f \cdot a$ of $A \otimes_{C_0(T)} C_0(T)$ onto A is equivariant and $C_0(T)$-linear. $\square$

Remark 3.4. If $G = \{ e \}$, we write $\mathrm{Br}(T)$ for $\mathrm{Br}_G(T)$. It is well-known that the map sending $[A]$ to $\delta(A)$ defines a bijection of $\mathrm{Br}(T)$ with $H^3(T;\mathbb{Z})$ (see, for example, [35, Theorem 3.5]). Proposition 2.2 implies that $[A] \mapsto \delta(A)$ is a (semi-group) isomorphism; in particular, $\mathrm{Br}(T)$ is a group.

If V is a complex vector space, then we will write $\overline{V}$ for the conjugate space: that is, $\overline{V}$ coincides with V as a set, and if $\flat = \flat_V : V \to \overline{V}$ is the identity map, then scalar multiplication on $\overline{V}$ is given by $\lambda \cdot \flat(v) = \flat(\bar{\lambda} \cdot b)$. In the event A is a C^*-algebra with spectrum T, then $\overline{A}$ is again a C^*-algebra[3] with spectrum T, and if $\phi \in C_0(T)$, then $\phi \cdot \flat_A(a) = \flat_A(\bar{\phi} \cdot a)$. Furthermore, if $\mathcal{X}$ is an $A\text{--}_T\,B$-imprimitivity bimodule, then $\overline{\mathcal{X}}$ is naturally an $\overline{A}\text{--}_T\,\overline{B}$-imprimitivity bimodule:

$$\flat_A(a) \cdot \flat_{\mathcal{X}}(x) = \flat_{\mathcal{X}}(a \cdot x) \qquad \flat_{\mathcal{X}}(x) \cdot \flat_B(b) = \flat_{\mathcal{X}}(x \cdot b)$$
$${}_{\overline{A}}\big\langle \flat_{\mathcal{X}}(x), \flat_{\mathcal{X}}(y) \big\rangle = \flat_A\big({}_A\langle x, y\rangle\big) \qquad \big\langle \flat_{\mathcal{X}}(x), \flat_{\mathcal{X}}(y) \big\rangle_{\overline{B}} = \flat_B\big(\langle x, y\rangle_B\big).$$

Of course, if (A, α) is in $\mathfrak{Br}_G(T)$, then so is $(\overline{A}, \bar{\alpha})$, where $\bar{\alpha}_s\big(\flat(a)\big) = \flat\big(\alpha_s(a)\big)$.

Remark 3.5. If $\mathcal{X}$ is a $A\text{--}_T\,C_0(T)$-imprimitivity bimodule, then we will view $\mathcal{X} \mathbin{\widehat{\otimes}}_{C_0(T)} \overline{\mathcal{X}}$ as a $A \otimes_{C_0(T)} \overline{A}\text{--}_T\,C_0(T)$-imprimitivity bimodule by identifying $C_0(T) \otimes_{C_0(T)} \overline{C_0(T)}$ with $C_0(T)$ via the isomorphism determined by $\phi \otimes \flat(\psi) \mapsto \phi\overline{\psi}$.

Theorem 3.6. *With the binary operation defined in* (3.2), $\mathrm{Br}_G(T)$ *is a group. The inverse of* $[A, \alpha]$ *is given by* $[\overline{A}, \bar{\alpha}]$.

Remark 3.7. The theorem has several immediate and interesting consequences. For example, we can reduce the problem of classifying G-actions on a given stable continuous-trace C^*-algebra A with spectrum T which cover the given action ℓ on T to (1) finding an *single* action α on A covering ℓ and (2) classifying all G-actions on $C_0(T, \mathcal{K})$ covering ℓ. To make this precise, observe that the homomorphism $F : \mathrm{Br}_G(T) \to \mathrm{Br}(T)$ defined by $F\big([A, \alpha]\big) = \delta(A)$ (called the *Forgetful Homomorphism*) has as its kernel exactly the subgroup of $\mathrm{Br}_G(T)$ consisting of classes (which have representatives) of the form $\big(C_0(T, K), \sigma\big)$. Then the assertion above is simply that the classes in $\mathrm{Br}_G(T)$ coming from actions on A are precisely those in $F^{-1}\big(\delta(A)\big) = [A, \alpha]\ker(F)$.

To prove Theorem 3.6, all that remains to be shown is the last assertion. This will require the remainder of the section. We fix (A, α) in $\mathfrak{Br}_G(T)$. As before, we can choose data $\{ F_i \}_{i \in I}$, $\{ \mathcal{X}_i \}$, $\{ g_{ij} \}$, and $\{ \nu_{ijk} \}$ as in Proposition 2.2. Naturally, we can define $\bar{g}_{ij} : \overline{\mathcal{X}}_J^{F_{ij}} \to \overline{\mathcal{X}}_i^{F_{ij}}$ by $\bar{g}_{ij}\big(\flat(x)\big) = \flat\big(g_{ij}(x)\big)$. Then we can produce data for $A \otimes_{C_0(T)} \overline{A}$ of the form $\{ F_i \}$, $\{ \mathcal{X}_i \mathbin{\widehat{\otimes}}_{C(F_i)} \overline{\mathcal{X}}_i \}$, and $\{ h_{ij} \}$, where $h_{ij} = g_{ij} \mathbin{\widehat{\otimes}} \bar{g}_{ij}$. Notice that

$$h_{ij}^{F_{ijk}} \circ h_{jk}^{F_{ijk}} = h_{ik}^{F_{ijk}}. \tag{3.3}$$

Using the cocycle property (3.3), we can construct a $A \otimes_{C_0(T)} \overline{A}\text{--}_T\,C_0(T)$-imprimitivity bimodule as in [30, 35]. Specifically, we set

$$\mathcal{Y}' = \{ (y_i) \in \prod_I \mathcal{X}_i \mathbin{\widehat{\otimes}}_{C(F_i)} \overline{\mathcal{X}}_i : h_{ij}\big(y_j^{F_{ij}}\big) = y_i^{F_{ij}} \}.$$

[3] In fact, $\overline{A}$ is $*$-isomorphic to the "opposite algebra" A^{op} via the map $x^{\mathrm{op}} \in A^{\mathrm{op}} \mapsto \flat(x^*) \in \overline{A}$.

From (3.3) we deduce that if $t \in F_{ij}$ and $x = (x_i), y = (y_i) \in \mathcal{Y}'$, then

$$ {}_{(A\otimes_{C_0(T)}\overline{A})^{F_i}}\langle x_i, y_i\rangle(t) = {}_{(A\otimes_{C_0(T)}\overline{A})^{F_j}}\langle x_j, y_j\rangle(t). $$

(Since $A \otimes_{C_0(T)} \overline{A}$ has Hausdorff spectrum T, we may view it as the section algebra of a C^*-bundle over T.) Since a similar equation holds for the $C(F_i)$-valued inner products, we obtain well-defined sesqui-linear forms on $\mathcal{Y}'$ by the formulas

$$ \begin{aligned} {}_{A\otimes_{C_0(T)}\overline{A}}\langle x, y\rangle(t) &= {}_{(A\otimes_{C_0(T)}\overline{A})^{F_i}}\langle x_i, y_i\rangle(t), \text{ and} \\ \langle x, y\rangle_{C_0(T)}(t) &= \langle x_i, y_i\rangle_{C(F_i)}(t), \end{aligned} \tag{3.4} $$

for $t \in F_i$. Notice that $\mathcal{Y}'$ admits natural left and right actions of $A \otimes_{C_0(T)} \overline{A}$ and $C_0(T)$, respectively. The next lemma can be proved along the lines of [30, Proposition 2.3].

Lemma 3.8. *With the inner products given by* (3.4)*,*

$$ \mathcal{Y} = \{\, y \in \mathcal{Y}' : t \mapsto \langle y, y\rangle_{C_0(T)}(t) \text{ vanishes at infinity} \,\} $$

is a complete $A \otimes_{C_0(T)} \overline{A}$ –$_T$ $C_0(T)$-imprimitivity bimodule.

While $\mathcal{Y}$ is the sort of module required in Theorem 3.6, it unfortunately carries no obvious G-action—let alone one equivalent to τ. To overcome this, we will want to see that $\mathcal{Y}$ is isomorphic to a special subalgebra of A. To do this let

$$ \mathfrak{N} = \{\, a \in A : t \mapsto \operatorname{tr}(a^*a)(t) \text{ is in } C_0(T) \,\}. $$

Then $\langle x, y\rangle_{C_0(T)}(t) = \operatorname{tr}(x^*y)(t)$ defines a $C_0(T)$-valued inner product on $\mathfrak{N}$ ([10, 4.5.2]). Because A has continuous trace, $\mathfrak{N}$ is dense in A by Definition 4.5.2 and Lemma 4.5.1(ii) of [10]; thus $\operatorname{span}\{\, \langle x, y\rangle_{C_0(T)} : x, y \in \mathfrak{N} \,\}$ is an ideal in $C_0(T)$ without common zeros, and hence is dense in $C_0(T)$. The next result is a pleasant surprise.

Lemma 3.9. *With respect to the norm $\|a\|_2 = \|\langle a, a\rangle_{C_0(T)}\|_\infty^{1/2}$, $\mathfrak{N}$ is a (full) right Hilbert $C_0(T)$-module.*

Proof. The only issue is to see that $\mathfrak{N}$ is complete. Observe that if $a \in \mathfrak{N}$, then $a(t)$ is a Hilbert-Schmidt operator, and $\|\cdot\|_2$ induces the Hilbert-Schmidt norm $\|\cdot\|_{\mathrm{HS}}$ on $\mathfrak{N}(t) = \{\, a(t) \in A(t) : a \in \mathfrak{N} \,\}$. In particular, for any $t \in T$, $\|a(t)\|_{\mathrm{HS}} \le \|a\|_2$. So suppose that $\{\, a_n \,\}$ is $\|\cdot\|_2$-Cauchy in $\mathfrak{N}$. Since the C^*-norm $\|\cdot\|$ is dominated by $\|\cdot\|_2$, we have a_n converging to some a in A. Since the Hilbert-Schmidt operators on any Hilbert space are complete in the Hilbert-Schmidt norm, the Cauchy sequence $a_n(t)$ must converge, and must converge to $a(t)$ in the Hilbert-Schmidt norm and $\operatorname{tr}(a^*a)(t) < \infty$.

We still have to show that $a \in \mathfrak{N}$ and that a_n converges to a in $\mathfrak{N}$. Fix $\epsilon > 0$. Choose N so that $n, m \ge N$ implies that $\|a_n - a_m\|_2 < \epsilon/2$. If $t \in T$, then there is a $k \ge N$ so that $\|a_k(t) - a(t)\|_{\mathrm{HS}} < \epsilon/2$. Then if $n \ge N$,

$$ \begin{aligned} \|a_n(t) - a(t)\|_{\mathrm{HS}} &\le \|a_n(t) - a_k(t)\|_{\mathrm{HS}} + \|a_k(t) - a(t)\|_{\mathrm{HS}} \\ &\le \|a_n - a_k\|_2 + \|a_k(t) - a(t)\|_{\mathrm{HS}} < \epsilon. \end{aligned} $$

Our result follows as $\|x\|_2 = \sup_t \|x(t)\|_{\mathrm{HS}}$. $\square$

We also need the following technical result. It is a special case of [12, Lemma 1].

Lemma 3.10 (Green). *If $x, y \in \mathfrak{X}_i$ and $t \in F_i$, then* $\operatorname{tr}\bigl({}_{A^{F_i}}\langle x, y\rangle(t)\bigr) = \langle y, x\rangle_{C(F_i)}(t)$.

Proof. This result is proved for $x = y$ in the second paragraph of the proof of Theorem 2.15 in [45]. Since everything in sight is trace-class, the general case follows from the usual polarization identities: $4\,{}_{A^{F_j}}\langle x, y\rangle = \sum_{k=0}^{3} {}_{A^{F_j}}\langle x + i^k y, x + i^k y\rangle$ and $4\langle x, y\rangle_{C_0(T)} = \sum_{k=0}^{3}\langle i^k x + y, i^k x + y\rangle_{C_0(T)}$. $\square$

We define a map $\Phi_i : \mathfrak{X}_i \mathbin{\widehat{\otimes}}_{C(F_i)} \overline{\mathfrak{X}}_i \to \mathfrak{N}^{F_i}$ as follows. Suppose $y_i = \sum_k x_k \otimes \flat(z_k)$ is a sum of elementary tensors in $\mathfrak{X}_i \odot \overline{\mathfrak{X}}_i$. Then for $t \in F_i$,

$$\Phi_i(y_i)(t) = \sum_k {}_{A^{F_i}}\langle x_k, z_k\rangle(t),$$

defines a map on $\mathfrak{X}_i \odot \overline{\mathfrak{X}}_i$ (it is sesqui-linear), which preserves inner products by the following computation. Let $y_i' = \sum_k u_k \otimes \flat(v_k)$. Then

$$\begin{aligned}
\bigl\langle \Phi_i(y_i), \Phi_i(y_i')\bigr\rangle_{C(F_i)} &= \operatorname{tr}\bigl(\Phi_i(y_i)^*\Phi(y_i')(t)\bigr) \\
&= \operatorname{tr}\Bigl(\sum_{k,r} {}_{A^{F_i}}\langle x_k, z_k\rangle^*\, {}_{A^{F_i}}\langle u_r, v_r\rangle\Bigr)(t) \\
&= \operatorname{tr}\Bigl(\sum_{r,k} {}_{A^{F_i}}\bigl\langle z_k \cdot \langle x_k, u_r\rangle_{C(F_i)}, v_r\bigr\rangle\Bigr)(t)
\end{aligned} \tag{3.5}$$

which, using Lemma 3.10, is

$$\begin{aligned}
&= \sum_{r,k}\bigl\langle v_r, z_k \cdot \langle x_k, u_r\rangle_{C(F_i)}\bigr\rangle_{C(F_i)}(t) \\
&= \langle y_i, y_i'\rangle_{C(F_i)}(t)
\end{aligned}$$

Thus, Φ_i extends to a map on $\mathfrak{X}_i \mathbin{\widehat{\otimes}}_{C(F_i)} \overline{\mathfrak{X}}_i$ taking values in $\mathfrak{N}^{F_i}$ since the latter is complete. Notice that we may replace $\mathfrak{X}_i$ by $\mathfrak{X}_i^{F_{ij}}$ in the above to obtain a similar map $\Phi_i^{F_{ij}}$ into $\mathfrak{N}^{F_{ij}}$, and that for $t \in F_{ij}$ we have $\Phi_i(y)(t) = \Phi_i^{F_{ij}}(y^{F_{ij}})(t)$ for any $y \in \mathfrak{X}_i \mathbin{\widehat{\otimes}}_{C(F_i)} \overline{\mathfrak{X}}_i$. Now suppose that $y = (y_i) \in \mathfrak{Y}$, $t \in F_{ij}$, and $\epsilon > 0$. Choose $\tilde{y}_j \in \mathfrak{X}_j \odot \overline{\mathfrak{X}}_j$ so that $\|y_j - \tilde{y}_j\| < \epsilon$. Thus $\|y_j^{F_{ij}} - \tilde{y}_j^{F_{ij}}\| < \epsilon$, and (3.5) implies that $|\Phi_j(y_j)(t) - \Phi_j(\tilde{y}_j)(t)| < \epsilon$. As $y \in \mathfrak{Y}$, $\|y_i^{F_{ij}} - h_{ij}(\tilde{y}_j^{F_{ij}})\| < \epsilon$, and a calculation on elementary tensors shows that $\Phi_i\bigl(h_{ij}(\tilde{y}_j)\bigr)(t) = \Phi_j(\tilde{y}_j^{F_{ij}})(t)$. It follows that $|\Phi_i(y_i)(t) - \Phi_j(y_j)(t)| < \epsilon$; since ϵ was arbitrary, we have $\Phi_i(y_i)(t) = \Phi_j(y_j)(t)$ Thus we can define $\Phi : \mathfrak{Y} \to \mathfrak{N}$ by setting $\Phi\bigl((y_i)\bigr)(t) = \Phi_j(y_j)(t)$ for $t \in F_j$. We have shown above that this is well-defined, and it follows from (3.5) that Φ does indeed take values in $\mathfrak{N}$.

Proposition 3.11. *The map Φ defined above extends to a Hilbert $C_0(T)$-module isomorphism from $\mathfrak{Y}$* onto $\mathfrak{N}$.

Proof. Since we have already shown that Φ preserves inner products, and Φ is clearly $C_0(T)$-linear, we only have to show that $\Phi(\mathcal{Y})$ is dense in $\mathfrak{N}$. Observe that the $C_0(T)$-submodule $\Phi(\mathcal{Y})$ is also an ideal in A: $a\Phi(y) = \Phi\big((a \otimes 1)\cdot y\big)$ and $\Phi(y)a = \Phi\big((1 \otimes \flat(a^*))\cdot y\big)$. Since $\Phi(\mathcal{Y})$ is certainly C^*-norm dense in A, we have that $\Phi(\mathcal{Y})(t)$ is norm dense in $A(t)$ for each $t \in T$. In particular, $\Phi(\mathcal{Y})(t)$ contains the finite-rank operators. (The finite-rank operators are the Pedersen ideal in $\mathcal{K}$ [26, §5.6].) Therefore $\Phi(\mathcal{Y})(t)$ is dense in $X(t)$ in the Hilbert-Schmidt norm. Now fix $a \in \mathfrak{N}$ and $\epsilon > 0$. Choose a compact set $C \subseteq T$ such that $\|a\|^2_{\mathrm{HS}} = |\operatorname{tr}(a^*a)(t)| < \epsilon/4$ if $t \notin C$. For each $t \in C$, there is a $y \in \mathcal{Y}$ such that $\|\Phi(y)(t) - a(t)\|_{\mathrm{HS}} < \epsilon/4$, and a relatively compact neighborhood U of t such that

$$\|\Phi(y)(t') - a(t')\|_{\mathrm{HS}} < \frac{\epsilon}{2} \text{ for all } t' \in U. \tag{3.6}$$

Next choose a partition of unity $\{\,\phi_i\,\}_{i=1}^n \subseteq C_c(T)$ subordinate to a cover $U_1, \dots, U_n$ of C as in (3.6) for elements $y_1, \dots, y_n$ in $\mathcal{Y}$. (That is, $\sum_i \phi_i \equiv 1$ on C and $\sum_i \phi_i \le 1$ otherwise.) Let $y = \sum y_i \cdot \phi_i$. Since $\|a - a \cdot \sum \phi_i\|_2 < \epsilon/2$, it follows that $\|\Phi(y) - a\|_2 < \epsilon$. This proves density and completes the proof. □

Corollary 3.12. *Let $\mathcal{K}(\mathfrak{N})$ denote the compact operators on the Hilbert $C_0(T)$-module $\mathfrak{N}$. (In less modern terms, $\mathcal{K}(\mathfrak{N})$ is the imprimitivity algebra of the right $C_0(T)$-rigged space $\mathfrak{N}$.) Then there is a $C_0(T)$-isomorphism Q of $A \otimes_{C_0(T)} \overline{A}$ onto $\mathcal{K}(\mathfrak{N})$ which satisfies*

$$Q\big(a \otimes \flat(b)\big)(c) = acb^*. \tag{3.7}$$

Proof. Recall from Lemma 3.8 that $A \otimes_{C_0(T)} \overline{A} \cong \mathcal{K}(\mathcal{Y})$. Define $Q : \mathcal{L}(\mathcal{Y}) \to \mathcal{L}(\mathfrak{N})$ by $Q(T)(x) = \Phi\big(T\big(\Phi^{-1}(x)\big)\big)$. Then check that $Q\big(\mathcal{K}(\mathcal{Y})\big) = \mathcal{K}(\mathfrak{N})$, and note that Q is $C_0(T)$-linear.

Finally, let $y = \big(x_i \otimes \flat(y_i)\big)$ be an element of $\mathcal{Y}$ whose components are elementary tensors, and let T be left-multiplication by $a \otimes \flat(b)$. Then elements of the form $\Phi(y)$ span a dense subset of $\mathfrak{N}$[4] and $\Phi\big(T(y)\big) = \Phi\big((ax_i \otimes \flat(by_i))\big) = a\Phi(y)b^*$. □

We now claim that, for each $s \in G$, $\alpha_s(\mathfrak{N}) = \mathfrak{N}$. By assumption on (A, α), if $a \in A$ and π_t denotes evaluation at t, then $\pi_{s^{-1}\cdot t}$ is equivalent to $\pi \circ \alpha_s$, and

$$\alpha_s(a^*a)(t) = \pi_t\big(\alpha_s(a^*a)\big) = V\pi_{s^{-1}\cdot t}(a^*a)V^* = Va^*a(s^{-1}\cdot t)V^*$$

for some unitary V. It follows that $\operatorname{tr}\big(\alpha_s(a^*a)(t)\big) = \operatorname{tr}\big(a^*a(s^{-1}\cdot t)\big)$ for all $a \in A$. Therefore if $x, y \in \mathfrak{N}$, then a polarization argument implies that

$$\operatorname{tr}\big(\alpha_s(x^*y)(t)\big) = \operatorname{tr}\big((x^*y)(s^{-1}\cdot t)\big), \tag{3.8}$$

satisfying the claim.

[4] To see this, note that $\mathfrak{N}(t)$ is a Hilbert space and a partition of unity argument shows that it is enough to see that the span of such elements is dense in each $\mathfrak{N}(t)$. Now the assertion follows from the following: (1) elementary tensors span a dense subset of $\mathcal{X}_i \mathbin{\widehat{\otimes}}_{C(F_i)} \overline{\mathcal{X}}_i$; (2) if $t \in F_i$, $\Phi(y)(t) = \Phi_i(y)(t)$; and (3) if $y_0 \in \mathcal{X}_i \odot \overline{\mathcal{X}}_i$, then the argument in [30, Lemma 2.2] shows that there is a $y = (y_i) \in \mathcal{Y}$ with $y_i = y_0$ and with the other $y_j \in \mathcal{X}_j \odot \overline{\mathcal{X}}_j$.

Proposition 3.13. *The action defined by $u_s(x) = \alpha_s(x)$ is strongly continuous on $\mathfrak{N}$ and satisfies*

$$\big\langle u_r(x), u_r(y)\big\rangle_{C_0(T)}(t) = \langle x, y\rangle_{C_0(T)}(r^{-1}\cdot t)$$

for all $x, y \in \mathfrak{N}$, $t \in T$, and $s \in G$.

Proof. The second assertion follows from (3.8). Thus, because we have already shown that $\alpha_s(\mathfrak{N}) = \mathfrak{N}$, we only have to show strong continuity.

We first claim that $\mathfrak{M} = \mathfrak{N}^2$ is $\|\cdot\|_2$-norm dense in $\mathfrak{N}$. (In the notation of [10], $\mathfrak{N}$ coincides with $\mathfrak{n}$.) By [10, 4.5.1], $\mathfrak{M}$ is C^*-norm dense in A, so $\mathfrak{M}(t)$ contains the finite-rank operators for each $t \in T$ and hence is dense in $\mathfrak{N}(t)$ in the $\|\cdot\|_{\mathrm{HS}}$-norm. Thus given $t_0 \in T$ and $\epsilon > 0$, there is a neighborhood U of t_0 and $b \in \mathfrak{M}$ such that $\|a(t) - b(t)\|_{\mathrm{HS}} < \epsilon/2$ for all $t \in U$. Another partition of unity argument as in Proposition 3.11 implies that there is a $b \in \mathfrak{M}$ such that $\|a - b\|_2 < \epsilon$; this establishes the claim.

Since each u_t is $\|\cdot\|_2$-isometric, it suffices to show that $\lim_{s\to e}\|\alpha_s(a) - a\|_2 = 0$; from the previous paragraph we can assume that $a \in \mathfrak{M}$. It follows from (3.8) that $a(t)$ and $\alpha_s(a)(t)$ are trace class operators. Recall that $\|T\|_1 = \operatorname{tr}\big(|T|\big)$ is a norm on the trace-class operators. While it is apparent that $\|a(t)\|_1 = \|\alpha_s(a)(s\cdot t)\|_1$ if $a \geq 0$ (see (3.8)), we do not know whether this holds in general as it is not obvious that $|a| \in \mathfrak{M}$ if a is. However, $a = \sum_{i=1}^n \alpha_i a_i$ where $\alpha_i \in \mathbb{C}$ and $a_i \in \mathfrak{M}^+$ by [10, 4.5.1]. Thus, $\|\alpha_s(a)(t)\|_1 \leq \sum_i |\alpha_i|\|\alpha_s(a_i)\|_1 = \sum_i |\alpha_i|\|a_i(s^{-1}\cdot t)\|_1$, and there is a constant K, which depends only on a (and not on $s \in G$), so that $\sup_t \|\alpha_s(a)(t)\|_1 \leq K$. Therefore, if $\|\cdot\|$ denotes the C^*-norm, then

$$\begin{aligned}\|\alpha_s(a) - a\|_2 &= \sup_t \|(\alpha_s(a) - a)^*(\alpha_s(a) - a)(t)\|_1\\ &\leq \sup_t \|\alpha_s(a) - a\|\,\|(\alpha_s(a) - a)(t)\|_1\\ &\leq \|\alpha_s(a) - a\|\big(K + \sup_t \|a(t)\|_1\big),\end{aligned}$$

where the second inequality follows from [27, 3.4.10]. The conclusion now follows from the strong continuity of α in the C^*-norm on A. □

Lemma 3.14 ([7, §3]). *Suppose that $\mathcal{X}$ is an A–B-imprimitivity bimodule and that $u : G \to \operatorname{Aut}(\mathcal{X})$ is an action of G on $\mathcal{X}$: that is, there is a strongly continuous automorphism $\tau : G \to \operatorname{Aut}(B)$ such that $\tau_s\big(\langle x, y\rangle_B \cdot b\big) = \big\langle u_s(x), u_s(y)\big\rangle_B \cdot \tau_s(b)$. Then, if $T \in \mathcal{L}(\mathcal{X})$,*

$$\alpha_s(T) = u_s T u_s^{-1} \tag{3.9}$$

is in $\mathcal{L}(\mathcal{X})$, and (3.9) defines a strongly continuous automorphism group $\alpha : G \to \operatorname{Aut}(A)$ satisfying $\alpha_s\big(a \cdot {}_A\langle x, y\rangle\big) = \alpha_s(a) \cdot {}_A\big\langle u_s(x), u_s(y)\big\rangle$.

Proof. Certainly, α_s defines an automorphism of $\mathcal{L}(\mathcal{X})$, and since $u_s\langle x, y\rangle u_s^{-1} = {}_A\big\langle u_s(x), u_s(y)\big\rangle$, α_s restricts to an automorphism of A. The rest is straightforward. □

Proof of Theorem 3.6. We have shown above that $(\mathfrak{N}, u)$ is a Morita equivalence between the systems $(C_0(T), \tau)$ and $(A \otimes_{C_0(T)} \overline{A}, \beta)$ where $\beta_s(a\otimes\flat(b)) = u_s\big(Q\big(a\otimes\flat(b)\big)\big)u_s^{-1}$. Now let $c \in \mathfrak{N}$ and recall that u_t is the restriction of α_t to $\mathfrak{N}$. Then

$$\begin{aligned}\beta_t\big(a \otimes \flat(b)\big) \cdot c &= \alpha_t\big(Q\big(a \otimes \flat(b)\big)\big(\alpha_t^{-1}(c)\big)\big) = \alpha_t(a)c\alpha_t(b)^* \\ &= Q\big(\alpha_t(a) \otimes \bar{\alpha}_t(\flat(b))\big)(c).\end{aligned}$$

Thus, $\beta = \alpha \otimes \bar{\alpha}$, and we have shown that $[A, \alpha]^{-1}$ exists and equals $[\overline{A}, \bar{\alpha}]$. This completes the proof. □

4. The Forgetful Homomorphism

In this section we will require that $H^2(T; \mathbb{Z})$ be countable, and as before, that (G, T) be a second countable locally compact transformation group. The homomorphism $F : \mathrm{Br}_G(T) \to \mathrm{Br}(T)$ defined by $F\big([A, \alpha]\big) = \delta(A)$ is called the *Forgetful Homomorphism* (where we identify $\mathrm{Br}(T)$ with $H^3(T; \mathbb{Z})$ as in Remark 3.4). The image of F is of considerable interest as it describes exactly which stable algebras admit actions inducing a given action on T. More precisely, we have the following lemma.

Lemma 4.1. *If A is a stable, separable continuous-trace C^*-algebra with spectrum T, then A admits an an automorphism group $\alpha : G \to \mathrm{Aut}(A)$ inducing the given action on T if and only if $\delta(A)$ is in $\mathrm{Im}(F)$.*

Proof. From the definitions it is clear that $\delta(A)$ is in the image of F if and only if A is (strongly) Morita equivalent over T to an algebra B which admits an action α covering τ; i.e., $(B, \alpha) \in \mathfrak{Br}_G(T)$. But then $(B \otimes \mathcal{K}, \alpha \otimes \mathrm{id}) \in \mathfrak{Br}_G(T)$, and $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ by [4], and we have to check that this isomorphism is $C_0(T)$-linear. But since $\mathcal{X}$ is an $A\text{–}_T\,B$-bimodule, there is a natural action of $C_0(T)$ on the linking algebra C; since the isomorphism of $B \otimes \mathcal{K}$ and $A \otimes \mathcal{K}$ with $C \otimes \mathcal{K}$ are obtained by conjugation by partial isometries in $\mathcal{M}(C \otimes \mathcal{K})$ [4, §2], they are $C_0(T)$-linear. Finally, if A is stable, then A is $C_0(T)$-isomorphic to $A \otimes \mathcal{K}$ by [29, Lemma 4.3]. This proves the lemma. □

As an example of the significance of these ideas, notice that [31, Theorem 4.12] implies that F is surjective when $G = \mathbb{R}$ and (G, T) is a principal $\mathbb{T}$-bundle (provided, say, T/G is a CW-complex). The analysis of this section will give a substantial generalization of this result. Our approach is to identify three obstructions to an element $\mathfrak{c} \in H^3(T; \mathbb{Z})$ being in $\mathrm{Im}(F)$, and then to show that the vanishing of these obstructions is sufficient (as well as necessary).

The first obstruction is that in order that $\mathfrak{c} \in H^3(T; \mathbb{Z})$ be in $\mathrm{Im}(F)$, we must have

$$\mathfrak{c} \in H^3(T; \mathbb{Z})^G = \{\, \mathfrak{c} \in H^3(T; \mathbb{Z}) : s \cdot \mathfrak{c} = \mathfrak{c} \text{ for all } s \in G \,\} \tag{4.1}$$

(Here and in the sequel, we view $H^n(T; \mathbb{Z})$ as a G-module via the G-action on T; that is, if ℓ_s denotes the homeomorphism $t \mapsto s \cdot t$ of T, then $s \cdot \mathfrak{c} = \ell_{s^{-1}}^*(\mathfrak{c})$.) The necessity of (4.1) is a consequence of the following lemma which, although we present a different proof here, is contained in Theorem 2.22 of [28].

Lemma 4.2. *Suppose that A is a continuous-trace C^*-algebra with spectrum T, that $\alpha \in \operatorname{Aut}(A)$, and that h is the homeomorphism of T induced by α. Then $h^*\big(\delta(A)\big) = \delta(A)$.*

Proof. Let $h^*(A) = C_0(T) \otimes_{C_0(T)} A$ be the pull-back of A along $h : T \to T$. Then $\delta\big(h^*(A)\big) = h^*\big(\delta(A)\big)$ by [33, Proposition 1.4(1)]. On the other hand $\phi \otimes a \mapsto \phi \otimes \alpha(a)$ extends to a $C_0(T)$-isomorphism of $\mathrm{id}^*(A)$ onto $h^*(A)$. The result follows. □

Our other obstructions are obtained by identifying a subgroup of the Moore group $H^3\big(G, C(T, \mathbb{T})\big)$ and defining homomorphisms d_2 of $H^3(T;\mathbb{Z})^G$ into $H^2\big(G, H^2(T;\mathbb{Z})\big)$ and d_3 from the kernel of d_2 to the corresponding quotient of $H^3\big(G, C(T,\mathbb{T})\big)$. We then show that $\mathfrak{c} \in \operatorname{Im}(F)$ if and only if $\mathfrak{c} \in H^3(T;\mathbb{Z})^G$ and $d_2(\mathfrak{c}) = d_3(\mathfrak{c}) = 0$. This will occupy the remainder of this section.

The basic idea for the construction of d_2 is as follows. Let $\ell : G \to \operatorname{Homeo}(T)$ be the map induced by (G, T). Notice that if $\mathfrak{c} \in H^3(T;\mathbb{Z})^G$, then $\ell(G) \subseteq \operatorname{Homeo}_{\mathfrak{c}}(T)$. If A is the essentially unique *stable* continuous-trace C^*-algebra with $\delta(A) = \mathfrak{c}$, then by [28, Theorem 2.22] there is a short exact sequence

$$1 \longrightarrow \operatorname{Aut}_{C_0(T)}(A)/\operatorname{Inn}(A) \longrightarrow \operatorname{Out}(A) \longrightarrow \operatorname{Homeo}_{\mathfrak{c}}(T) \longrightarrow 1.$$

Furthermore, it follows from [28, Theorem 2.1] that there is an isomorphism $\zeta : \operatorname{Aut}_{C_0(T)}(A)/\operatorname{Inn}(A) \to H^2(T;\mathbb{Z})$. Therefore there should be an obstruction $d_2(\mathfrak{c})$ in $H^2\big(G, H^2(T;\mathbb{Z})\big)$ to lifting ℓ to a homomorphism $\gamma : G \to \operatorname{Out}(A)$.

The existence of d_2 will follow from the next lemma. Notice that if N is a normal *abelian* subgroup of a group H, then H/N acts on N by conjugation.

Lemma 4.3. *Suppose that H is a Polish group, that N is a closed normal abelian subgroup, and that $\ell : G \to H/N$ is a continuous homomorphism. Then N is a G-module (where $g \in G$ acts on $n \in N$ by $g \cdot n = \ell_g n \ell_g^{-1}$), and there is a cohomology class $\mathfrak{c}(\ell) \in H^2(G, N)$ which vanishes if and only if there is a continuous homomorphism $\gamma : G \to H$ which lifts ℓ (i.e., $\gamma_g N = \ell_g$). In fact, one obtains a cocycle $n \in Z^2(G, N)$ representing $\mathfrak{c}$ by taking any Borel lift γ' of ℓ, and defining n by*

$$\gamma'_s \gamma'_t = n(s,t) \gamma'_{st}. \tag{4.2}$$

Proof. By [20, Proposition 4] we can find a Borel section $s : H/N \to H$ for the quotient map such that $s(N) = e$. Define $\gamma' = s \circ \ell$, and define n by (4.2). Then n is certainly Borel and comparing $\gamma'_r(\gamma'_s\gamma'_t)$ with $(\gamma'_r\gamma'_s)\gamma'_t$ shows that n is indeed a cocycle. A standard argument shows that the class of n in $H^2(G, N)$ is independent of our choice of section s.

Evidently, if γ' is a homomorphism, then $[n] = 0$ as n is identically equal to e. On the other hand, if $[n] = 0$, then there is a Borel function $\lambda : G \to N$ such that $\lambda_e = e$ and $n(s,t) = (s \cdot \lambda_t)^{-1} \lambda_s^{-1} \lambda_{st}$, and

$$(\lambda_s \gamma'_s)(\lambda_t \gamma'_t) = \lambda_s (s \cdot \lambda_t) \gamma'_s \gamma'_t = \lambda_s (s \cdot \lambda_t) n(s,t) \gamma'_{st} = \lambda_{st} \gamma'_{st}.$$

Therefore $\gamma = \lambda\gamma'$ is a Borel homomorphism lifting ℓ, which is continuous by [20, Proposition 5]. □

In order to apply Lemma 4.3 we have to see that the groups involved are Polish. However, because A is a separable C^*-algebra, then $\operatorname{Aut}(A)$, with the topology of pointwise convergence, is a Polish group. Then, as $\operatorname{Aut}_{C_0(T)}(A)$ is closed in $\operatorname{Aut}(A)$, it is also Polish. Since we are assuming that $H^2(T;\mathbb{Z})$ is countable, $\operatorname{Inn}(A)$ is open in $\operatorname{Aut}_{C_0(T)}(A)$ and closed in $\operatorname{Aut}(A)$ by [31, Theorem 0.8]. In particular, $\operatorname{Out}(A)$ is Polish, as is $H^2(T;\mathbb{Z}) \cong \operatorname{Aut}_{C_0(T)}(A)/\operatorname{Inn}(A)$ which is even discrete. Finally we give $\operatorname{Homeo}(T)$ the compact open topology. Then it is not hard to see that the map $\rho : \operatorname{Aut}(A) \to \operatorname{Homeo}(T)$ is continuous. In fact, $\operatorname{Homeo}(T)$ is homeomorphic to $\operatorname{Aut}\bigl(C_0(T)\bigr)$, and so is certainly a Polish group as well. We have not been able to show that $\operatorname{Homeo}_{\delta(A)}(T)$ is closed in $\operatorname{Homeo}(T)$, so we don't know for sure that it is Polish. But, as it follows from [28, Theorem 2.22] that ρ defines a continuous injection $h : \operatorname{Aut}(A)/\operatorname{Aut}_{C_0(T)}(A) \to \operatorname{Homeo}(T)$ with image exactly $\operatorname{Homeo}_{\delta(A)}(T)$, Souslin's Theorem [1, Theorem 3.3.2] implies that $\operatorname{Homeo}_{\delta(A)}(T)$ is a Borel subset of $\operatorname{Homeo}(T)$ and h is a Borel isomorphism. Thus we can view ℓ as a Borel homomorphism of G into the Polish group $\operatorname{Out}(A)/H^2(T;\mathbb{Z})$, which is automatically continuous. We can now apply Lemma 4.3 to

$$\begin{tikzcd} 0 \arrow[r] & H^2(T;\mathbb{Z}) \arrow[r] & \operatorname{Out}(A) \arrow[r] & \operatorname{Out}(A)/H^2(T;\mathbb{Z}) \arrow[r] & 0 \\ & & & G \arrow[u, "\ell"'] \arrow[ul, dotted, "\alpha"] & \end{tikzcd}$$

to get the desired obstruction $d_2\bigl(\delta(A)\bigr)$ in $H^2\bigl(G, H^2(T;\mathbb{Z})\bigr)$. However there is a small point to check. The G-action on $H^2(T;\mathbb{Z})$ coming from Lemma 4.3 is that coming from the identification of $H^2(T;\mathbb{Z})$ with the subgroup $\operatorname{Aut}_{C_0(T)}(A)/\operatorname{Inn}(A)$ of $\operatorname{Out}(A)$ via the isomorphism ζ. The G-action, then, is induced by conjugation by elements of $\operatorname{Aut}(A)/\operatorname{Aut}_{C_0(T)} A \cong_h \operatorname{Homeo}_{\delta(A)}(T)$. Thus the action of $s \in G$ on $[\phi] \in \operatorname{Aut}(A)/\operatorname{Aut}_{C_0(T)} A$ is given by $s \cdot [\phi] = [\gamma_s \phi \gamma_s^{-1}]$ where $\rho(\gamma_s) = \ell(s)$. On the other hand, the usual action of $s \in G$ on $\mathfrak{c} \in H^2(T;\mathbb{Z})$ is given by $s \cdot \mathfrak{c} = \ell_{s^{-1}}^*(\mathfrak{c})$. It is a relief that these actions coincide.

Lemma 4.4. *Let* $\zeta : \operatorname{Aut}_{C_0(T)}(A)/\operatorname{Inn}(A) \to H^2(T;\mathbb{Z})$ *be the isomorphism from* [28, Theorem2.1]. *If* $\gamma \in \operatorname{Aut}(A)$ *and* $\phi \in \operatorname{Aut}_{C_0(T)}(A)$, *then*

$$\zeta\bigl([\gamma\phi\gamma^{-1}]\bigr) = h(\gamma^{-1})^*\bigl(\zeta[\phi]\bigr).$$

Proof. To define ζ, we follow [41, §5]. View A as the sections of a C^*-bundle ξ over T. Then ϕ is locally implemented by multipliers. Thus there are an open cover $\{ N_i \}$ of T and $u_i \in \mathcal{M}(A)$ such that $\phi(a)(x) = u_i(x)a(x)u_i^*(x)$ for $x \in N_i$. (Recall that u_i can be viewed as a field of multipliers in $\mathcal{M}(A(x))$ [18].) Then $\zeta(\phi)$ is represented by the cocycle $\{ \lambda_{ij} \}$ where $\lambda_{ij}(x)u_j(x) = u_i(x)$ for $x \in N_{ij}$.

Let u_i and λ_{ij} be as above. For notational convenience, let $y = h(\gamma)^{-1}(x)$. Define an isomorphism γ_x from $A(y) \to A(x)$ by $\gamma_x\bigl(a(y)\bigr) = \gamma(a)(x)$. Then for

$x \in h(\gamma)(N_i)$, we have

$$\begin{aligned}(\gamma\phi\gamma^{-1})(a)(x) &= \gamma_x\bigl(\phi\gamma^{-1}(a)(y)\bigr) \\ &= \gamma_x\bigl(u_i(y)\gamma^{-1}(a)(y)u_i(y)^*\bigr) \\ &= \operatorname{Ad}\bigl(\gamma_x\bigl(u_i(y)\bigr)\bigr)\bigl[\gamma_x\bigl(\gamma^{-1}(a)(y)\bigr)\bigr] \\ &= \operatorname{Ad}\bigl(\gamma_x\bigl(u_i(y)\bigr)\bigr)\bigl[a(x)\bigr] = \operatorname{Ad}\bigl(\gamma(u_i)(x)\bigr)[a(x)],\end{aligned}$$

so $\gamma\phi\gamma^{-1}$ is implemented over $h(\gamma)(N_i)$ by $v_i = \gamma(u_i)$. Therefore for $x \in h(\gamma)(N_{ij})$, we have

$$\lambda_{ij}(y)v_j(x) = \lambda_{ij}(y)\gamma(u_j)(x) = \gamma_x\bigl(\lambda_{ij}(y)u_j(y)\bigr) = \gamma_x\bigl(v_i(y)\bigr) = v_i(x).$$

Thus $\zeta(\gamma\phi\gamma^{-1})$ is represented by the cocycle $\{\, h(\gamma)(N_i), \lambda_{ij} \circ h(\gamma)^{-1} \,\}$, which also represents $h(\gamma^{-1})^*\bigl(\zeta(\phi)\bigr)$. This completes the proof. $\square$

Lemma 4.5. *The map d_2 defined above is a homomorphism from $H^3(T;\mathbb{Z})^G$ to $H^2\bigl(G, H^2(T;\mathbb{Z})\bigr)$.*

Proof. Let A and B be stable continuous-trace C^*-algebras with spectrum T. Since there is a Borel section for the quotient map from $\operatorname{Aut}(A)$ onto $\operatorname{Out}(A)$, we may assume that there are Borel maps $\gamma : G \to \operatorname{Aut}(A)$ and $\delta : G \to \operatorname{Aut}(B)$ so that the obstructions are determined, respectively, by Borel maps $n : G \times G \to \operatorname{Aut}_{C_0(T)}(A)$ and $m : G \times G \to \operatorname{Aut}_{C_0(T)} B$ such that $\gamma_s\gamma_t = n(s,t)\gamma_{st}$ and $\delta_s\delta_t = m(s,t)\delta_{st}$. (Here, m and n need not be cocycles—only their images in $\operatorname{Aut}_{C_0(T)}(A)/\operatorname{Inn}(A)$ and $\operatorname{Aut}_{C_0(T)}(B)/\operatorname{Inn}(B)$.) Since γ_s and δ_s induce the same homeomorphism of T, $\gamma_s \otimes \delta_s$ defines an automorphism of $C = A \otimes_{C_0(T)} B$ which also induces the same homeomorphism of T. Moreover,

$$(\gamma_s \otimes \delta_s)(\gamma_t \otimes \delta_t) = \bigl(n(s,t) \otimes m(s,t)\bigr)(\gamma_{st} \otimes \delta_{st}),$$

so $d_2\bigl(\delta(C)\bigr)$ is determined by the cocycle $[n \otimes m]$ in $H^2\bigl(G, \operatorname{Aut}_{C_0(T)}(C)/\operatorname{Inn}(C)\bigr)$. But under the isomorphism $\zeta_C : \operatorname{Aut}_{C_0(T)}(C)/\operatorname{Inn}(C) \to H^2(T;\mathbb{Z})$ we have

$$\zeta_C(n \otimes m) = \zeta_A(n) + \zeta_B(m)$$

by [29, Proposition 3.10]. Therefore, using Proposition 2.2, we have $d_2\bigl(\delta(A) + \delta(B)\bigr) = d_2\bigl(\delta(A)\bigr) + d_2\bigl(\delta(B)\bigr)$ as required. $\square$

We now turn to the definition of d_3. The main technical tool will be the following lemma. Here we will need the twisted actions of [6, 22, 23]: a twisted action of G on A is a pair (α, u) consisting of Borel maps $\alpha : G \to \operatorname{Aut}(A)$ and $u : G \times G \to \mathcal{UM}(A)$ satisfying the axioms of [22, Definition 2.1].

Lemma 4.6. *Suppose that A and B are separable continuous-trace C^*-algebras with spectrum T.*

1. *Suppose that $\gamma : G \to \operatorname{Out}(A)$ is a continuous homomorphism and that $C(T, \mathbb{T})$ has the G-module structure coming from the G-action induced by γ and the natural map $h : \operatorname{Out}(A) \to \operatorname{Homeo}(T)$. Then there is a class*

$d_A(\gamma)$ in $H^3\big(G, C(T,\mathbb{T})\big)$ which vanishes if and only if there is a twisted action, $\alpha : G \to \operatorname{Aut}(A)$ and $u : G \times G \to \mathcal{UM}(A)$, such that

$$\begin{tikzcd} \operatorname{Aut}(A) \arrow[r, "q"] & \operatorname{Out}(A) \\ & G \arrow[u, "\gamma"'] \arrow[ul, dotted, "\alpha"] \end{tikzcd}$$

commutes.

(2) *Suppose that $\epsilon : G \to \operatorname{Out}(B)$ satisfies $h_B \circ \epsilon = h_A \circ \gamma$. Then there is a continuous homomorphism $\gamma \otimes \epsilon : G \to \operatorname{Out}(A \otimes_{C_0(T)} B)$, and $d_{A\otimes_{C_0(T)}B}(\gamma \otimes \epsilon) = d_A(\gamma)d_B(\epsilon)$.*

(3) *The map $\alpha \mapsto \bar\alpha$ from $\operatorname{Aut}(A) \to \operatorname{Aut}(\overline{A})$, defined by $\bar\alpha\big(\flat(a)\big) = \flat\big(\alpha(a)\big)$, maps $\operatorname{Ad} u$ to $\operatorname{Ad}\flat(u)$, and hence induces an isomorphism of $\operatorname{Out}(A)$ onto $\operatorname{Out}(\overline{A})$. If $\gamma : G \to \operatorname{Out}(A)$ is a continuous homomorphism, and $\bar\gamma : G \to \operatorname{Out}(\overline{A})$ is the corresponding map, then $d_{\overline{A}}(\bar\gamma) = d_A(\gamma)^{-1}$.*

(4) *If $\phi : A \to B$ is a $C_0(T)$-isomorphism, then conjugation by ϕ induces a homomorphism $\operatorname{Ad}(\phi) : \operatorname{Out}(A) \to \operatorname{Out}(B)$ and $d_B\big(\operatorname{Ad}(\phi) \circ \gamma\big) = d_A(\gamma)$.*

Proof. Choose a Borel lifting α of γ such that $\alpha_e = \mathrm{id}$. (Recall that we are assuming that $H^2(T;\mathbb{Z})$ is countable so that $\operatorname{Out}(A)$ is Polish.) Since $q(\alpha_s\alpha_t) = q(\alpha_{st})$, there is a Borel map $i : G \times G \to \operatorname{Inn}(A)$ such that $\alpha_s\alpha_t = i(s,t)\alpha_{st}$, and $i(s,e) = i(e,t) = \mathrm{id}$ for all $s,t \in G$. Since $\mathcal{UM}(A) \to \operatorname{Inn}(A)$ is a surjective homomorphism of Polish groups, there is a Borel map $u : G \times G \to \mathcal{UM}(A)$ such that

$$\alpha_s\alpha_t = \operatorname{Ad}\big(u(s,t)\big) \circ \alpha_{st}, \text{ and } u(s,e) = u(e,t) = 1. \tag{4.3}$$

Since $(\alpha_r\alpha_s)\alpha_t = \alpha_r(\alpha_s\alpha_t)$, we must have

$$\operatorname{Ad}\big(u(r,s)u(rs,t)\big) = \operatorname{Ad}\big(\alpha_r\big(u(s,t)\big)u(r,st)\big). \tag{4.4}$$

Therefore there is a Borel function $\lambda : G \times G \times G \to \mathcal{UZM}(A) = C(T,\mathbb{T})$ such that

$$\lambda(r,s,t)u(r,s)u(rs,t) = \alpha_r\big(u(s,t)\big)u(r,st). \tag{4.5}$$

Clearly, $\lambda(e,s,t) = \lambda(s,e,t) = \lambda(s,t,e) = 1$ for all $s,t \in G$. We want to show that λ is a 3-cocycle for the action of G on $C(T,\mathbb{T})$ defined above. Notice that, because α_s lifts γ_s, if we view $C(T,\mathbb{T})$ as the center of $\mathcal{UM}(A)$, then the action of G on $C(T,\mathbb{T})$ is given by $s \cdot f = \alpha_s(f)$. Our computations follow [19, §IV.8]. Let $L = \alpha_p\big(\alpha_r\big(u(s,t)\big)u(r,st)\big)u(p,rst)$. Then on the one hand

$$\begin{aligned} L &= \alpha_p\big(\lambda(r,s,t)u(r,s)u(r,st)\big)u(p,rst) \\ &= \alpha_p\big(\lambda(r,s,t)\big)\big[\lambda(p,r,s)u(p,r)u(pr,s)u(p,rs)^*\big] \\ &\qquad \big[\lambda(p,rs,t)u(p,rs)u(prs,t)u(p,rst)^*\big]u(p,rst) \\ &= \alpha_p\big(\lambda(r,s,t)\big)\lambda(p,r,s)\lambda(p,rs,t)u(p,r)u(pr,s)u(prs,t), \end{aligned}$$

while on the other hand,

$$\begin{aligned}L &= u(p,r)\alpha_{pr}\big(u(s,t)\big)u(p,r)^*\alpha_p\big(u(r,st)\big)u(p,rst)\\&= u(p,r)\big[\lambda(pr,s,t)u(pr,s)u(prs,t)u(pr,st)^*\big]u(p,r)^*\\&\qquad\big[\lambda(p,r,st)u(p,r)u(pr,st)u(p,rst)^*\big]u(p,rst)\\&= \lambda(pr,s,t)\lambda(p,r,st)u(p,r)u(pr,s)u(prs,t).\end{aligned}$$

It follows that λ is a 3-cocycle.

Next we observe that the class of λ depends only on γ and not on our choice of u or α. First suppose that v is another unitary-valued Borel map on $G\times G$ such that $\operatorname{Ad} u = \operatorname{Ad} v$. Then there is a Borel function $w : G\times G\to C(T,\mathbb{T})$ such that $v(s,t) = w(s,t)u(s,t)$. Let μ be the 3-cocycle corresponding to v as in (4.5). Then

$$\mu(r,s,t)w(r,s)w(rs,t) = \alpha_r\big(w(s,t)\big)w(r,st)\lambda(r,s,t),$$

so λ and μ define the same class in $H^3\big(G, C(T,\mathbb{T})\big)$. If β is another lift of γ, then there is a unitary valued Borel function $\tilde v$ on G such that $\beta_s = \operatorname{Ad}(\tilde v_s)\circ\alpha_s$. Then we can choose the lift

$$v(s,t) = \tilde v_s\alpha_s(\tilde v_t)u(s,t)\tilde v_{st}^*,$$

so that $\beta_s\beta_t = \operatorname{Ad}\big(v(s,t)\big)\beta_{st}$, and compute that

$$\begin{aligned}\beta_r\big(v(s,t)\big)v(r,st) &= \tilde v_r\alpha_r\big(\tilde v_s\alpha_s(\tilde v_t)u(s,t)\tilde v_{st}^*\big)\tilde v_r^*\tilde v_r\alpha_r(\tilde v_{st})u(r,st)\tilde v_{rst}^*\\&= \tilde v_r\alpha_r(\tilde v_s)\alpha_r\big(\alpha_s(\tilde v_t)\big)\alpha_r(u(s,t)\big)u(r,st)\tilde v_{rst}^*\\&= \tilde v_r\alpha_r(\tilde v_s)\big[u(r,s)\alpha_{rs}(\tilde v_t)u(r,s)^*\big]\big[\lambda(r,s,t)u(r,s)u(rs,t)\big]\tilde v_{rst}^*\\&= \lambda(r,s,t)\tilde v_r\alpha_r(\tilde v_s)u(r,s)\tilde v_{rs}^*\tilde v_{rs}\alpha_{rs}(\tilde v_t)u(rs,t)\tilde v_{rst}^*\\&= \lambda(r,s,t)v(r,s)v(rs,t).\end{aligned}$$

Thus we get the same cocycle λ for β provided we choose v as above; but since we have already observed that the class of λ in independent of our choice of v, we can conclude that the class $d_A(\gamma)$ depends only on γ, as claimed.

If $d_A(\gamma) = 0$, then $\lambda = \partial\mu$, and we can replace u by μu. (Then, of course, $\operatorname{Ad} u$ is unchanged and the corresponding λ is identically one.) Then it follows from (4.3) and (4.5) that (α, u) is a twisted action. On the other hand, if there is a twisted action (α, u), then the cocycle is certainly trivial. This proves (1).

Let γ, α, u, and λ be as above, and choose β lifting ϵ as well as v and μ in analogy with (4.3) and (4.5). Since $h_A\circ\gamma = h_B\circ\epsilon$, $\alpha\otimes\beta$ defines a Borel map into $\operatorname{Aut}(A\otimes_{C_0(T)} B)$. Moreover,

$$(\alpha_s\otimes\beta_s)(\alpha_t\otimes\beta_t) = \operatorname{Ad}\big(u(s,t)\otimes v(s,t)\big)\circ(\alpha_{st}\otimes\beta_{st}) \tag{4.6}$$

and with $w(s,t) = u(s,t)\otimes v(s,t)$, we have

$$(\alpha_r\otimes\beta_r)\big(w(s,t)\big)w(r,st) = \lambda\mu(r,s,t)w(r,s)w(rs,t). \tag{4.7}$$

Then (4.6) implies that $\alpha\otimes\beta$ defines a Borel, hence continuous, homomorphism $\gamma\otimes\epsilon$ of G into $\operatorname{Out}(A\otimes_{C_0(T)} B)$, satisfying $h_{A\otimes_{C_0(T)}B}\circ\gamma\otimes\epsilon = h_A\circ\gamma = h_B\circ\epsilon$, and (4.6) and (4.7) together imply that $d_{A\otimes_{C_0(T)}B}(\gamma\otimes\epsilon) = d_A(\gamma)d_B(\epsilon)$. This proves (2).

Part (3) is easy: $\mathcal{M}(\overline{A})$ is naturally isomorphic to $\overline{\mathcal{M}(A)}$, $\bar{\alpha}$ is a lift of $\bar{\gamma}$, and $\flat(u)$ satisfies $\bar{\alpha}_s\bar{\alpha}_t = \mathrm{Ad}\big(\flat\big(u(s,t)\big)\big)\bar{\alpha}_{st}$. But by definition of $\overline{A}$, applying $\flat$ to (4.5) replaces λ by $\bar{\lambda}$, which is the inverse of λ in H^3.

Finally, if α is a lift of γ, then $\beta_s = \phi\circ\alpha_s\circ\phi^{-1}$ is a lift of $\mathrm{Ad}(\phi)\circ\gamma$. But then $\beta_s\beta_t = \mathrm{Ad}\big(\phi\big(u(s,t)\big)\big)[\beta_{st}]$. Since ϕ is $C_0(T)$-linear, the obstruction to $\phi\big(u(\cdot,\cdot)\big)$ being a cocycle is that same as that for u. That completes the proof of the lemma. $\square$

To define $d_2' : H^1\big(G, H^2(T;\mathbb{Z})\big) \to H^3\big(G, C(T,\mathbb{T})\big)$ we apply the above lemma to $(A,\alpha) = \big(C_0(T,\mathcal{K}),\tau\big)$. Recall that $\zeta = \zeta_A : \mathrm{Aut}_{C_0(T)}(A)/\mathrm{Inn}(A) \to H^2(T;\mathbb{Z})$ is a G-equivariant isomorphism (Lemma 4.4). Thus, if $\rho\in Z^1\big(G, H^2(T;\mathbb{Z})\big)$, then $\rho_s' = \zeta^{-1}(\rho_s)\circ\tau_s$ defines a continuous map ([20, Theorem 3]) of G into $\mathrm{Out}(A)$ such that $h\circ\rho' = \ell$. (We will abuse notation slightly and use $\zeta^{-1}(\rho)$ to denote a representative in $\mathrm{Aut}_{C_0(T)}(A)$ of the class $\zeta^{-1}(\rho)$ in $\mathrm{Aut}_{C_0(T)}(A)/\mathrm{Inn}(A)$.) Thus, Lemma 4.6(1) gives us a class $d_A(\rho')$ in $H^3\big(G, C(T,\mathbb{T})\big)$ with the action of G on $C(T,\mathbb{T})$ being the expected one: $s\cdot f(t) = f\big(\ell_{s^{-1}}(t)\big) = f(s^{-1}\cdot t)$. If $\rho,\sigma\in Z^1\big(G, H^2(T;\mathbb{Z})\big)$, then as $h\circ\rho' = h\circ\sigma'$, part (2) of our lemma implies that $d_{A\otimes_{C_0(T)}A}(\rho'\otimes\sigma') = d_A(\rho')d_A(\sigma')$. But there is a $C_0(T)$-isomorphism ϕ of $A\otimes_{C_0(T)} A$ onto A. Therefore

$$\begin{aligned} d_{A\otimes_{C_0(T)}A}(\rho'\otimes\sigma') &= d_A\big(\phi\circ(\rho'\otimes\sigma')\circ\phi^{-1}\big) \\ &= d_A(\phi\circ\zeta^{-1}_{A\otimes_{C_0(T)}A}(\rho\sigma)\circ\phi^{-1}) \\ &= d_A(\zeta_A^{-1}\circ\rho\sigma) = d_A\big((\rho\sigma)'\big); \end{aligned}$$

the second equality is a consequence of [29, Proposition 3.10], the third follows from the general fact that if $\phi: A\to B$ is a $C_0(T)$-isomorphism of continuous-trace C^*-algebras, then $\zeta_B\big(\mathrm{Ad}(\phi)[\theta]\big) = \zeta_A(\theta)$, which results from the observation that if θ is locally implemented by $w_i\in\mathcal{M}(A)$, then $\mathrm{Ad}(\phi)[\theta]$ is implemented by $\phi(w_i)$.

Thus $\rho\mapsto d_A(\rho')$ defines a homomorphism from $Z^1\big(G, H^2(T;\mathbb{Z})\big)$ into the Moore group $H^3\big(G, C(T,\mathbb{T})\big)$. If $[\rho] = 0$ in H^1, then there is an element $b\in H^2(T;\mathbb{Z})$ such that $\rho_s = b(s\cdot b)^{-1}$. We can lift b to an element $\theta\in\mathrm{Aut}_{C_0(T)}\big(C_0(T,\mathcal{K})\big)$, and define $\alpha_s = \theta\tau_s\theta^{-1}$. Then

$$\alpha_s\alpha_t = (\theta\tau_s\theta^{-1})(\theta\tau_t\theta^{-1}) = \theta\tau_{st}\theta^{-1} = \alpha_{st}.$$

Thus ρ' lifts to a (trivially) twisted action, our homomorphism factors through $H^1\big(G, H^2(T;\mathbb{Z})\big)$, and we can define the required homomorphism d_2' by $d_2'\big([\rho]\big) = d_A(\rho')$.

Now suppose that $\mathfrak{c}\in H^3(T;\mathbb{Z})^G$ is in the kernel of d_2. If A is a stable continuous-trace C^*-algebra with spectrum T and with $\delta(A) = \mathfrak{c}$, then $d_2(\mathfrak{c}) = 0$ implies that there is a homomorphism $\gamma: G\to\mathrm{Out}(A)$ lifting the canonical map $\ell: G\to \mathrm{Homeo}_{\mathfrak{c}}(T)$ (i.e., $h\circ\gamma = \ell$). Then Lemma 4.6(1) gives us an obstruction $d_A(\gamma)$. If $\delta: G\to\mathrm{Out}(A)$ also satisfies $h\circ\delta = \ell$, then we claim that $d_A(\gamma)d_A(\delta)^{-1}$ belongs to $\mathrm{Im}(d_2')$. To see this, first recall that, as pointed out in the beginning of this section, there are $C_0(T)$-isomorphisms $\phi_1 : A\otimes_{C_0(T)}\overline{A}\to A\otimes_{C_0(T)}\overline{A}\otimes\mathcal{K}$ and $\phi_2 : A\otimes_{C_0(T)}\overline{A}\otimes\mathcal{K}\to C_0(T,\mathcal{K})$. (We have already seen that $\delta(A\otimes_{C_0(T)}\overline{A}) = 0$.)

Thus,

$$\begin{aligned} d_A(\gamma)d_A(\delta)^{-1} &= d_A(\gamma)d_{\overline{A}}(\bar\delta) \\ &= d_{A\otimes_{C_0(T)}\overline{A}}(\gamma\otimes\bar\delta) \\ &= d_{A\otimes_{C_0(T)}\overline{A}\otimes\mathcal{K}}\big(\mathrm{Ad}(\phi_1)[\gamma\otimes\bar\delta]\big) \\ &= d_{C_0(T,\mathcal{K})}\big(\mathrm{Ad}(\phi_2\phi_1)[\gamma\otimes\bar\delta]\big), \end{aligned}$$

which is by definition $d_2'(\rho)$ where $\rho_s = \zeta\big(\mathrm{Ad}(\phi_1\phi_2)[\gamma\otimes\bar\delta]\circ\tau_s^{-1}\big)$. This establishes the claim. Consequently, we may make the following definition of d_3.

Definition 4.7. Given $\mathfrak{c}\in\ker d_2$, then $d_3(\mathfrak{c})$ is defined to be the class of $d_A(\gamma)$ in $H^3\big(G,C(T,\mathbb{T})\big)$ modulo the image of d_2', where A is a stable continuous-trace C^*-algebra with spectrum T such that $\delta(A)=\mathfrak{c}$, and γ is any lift of the canonical map $\ell: G\to\mathrm{Homeo}_{\mathfrak{c}}(T)$ to a homomorphism $\gamma: G\to\mathrm{Out}(A)$.

Notice that it follows from Lemma 4.6(2) and Proposition 2.2 that d_3 is a homomorphism. Now we are ready to identify the kernel of d_3 with $\mathrm{Im}(F)$.

Lemma 4.8. *Suppose that $\mathfrak{c}\in H^3(T;\mathbb{Z})^G$. Then $\mathfrak{c}$ is in the kernel of both d_2 and d_3 if and only if there there is a twisted action $\alpha: G\to\mathrm{Aut}(A)$, $u: G\times G\to\mathcal{UM}(A)$ such that $h(\alpha_s)=\ell_s$ for all $s\in G$.*

Proof. We prove only the non-trivial direction. Since $d_2(\mathfrak{c})=0$, there is a homomorphism $\gamma: G\to\mathrm{Out}(A)$ such that $h\circ\gamma=\ell$, where A is a stable, continuous-trace C^*-algebra with spectrum T and $\delta(A)=\mathfrak{c}$. Since $d_3(\mathfrak{c})=0$, $d_A(\gamma)\in\mathrm{Im}(d_2')$. Thus we can choose $\rho\in Z^1\big(G,H^2(T;\mathbb{Z})\big)$ with $d_2'(\rho)=d_A(\gamma)^{-1}$. Since A is stable, A is $C_0(T)$-isomorphic to $A\otimes\mathcal{K}$ by [29, Lemma 4.3], and there is a $C_0(T)$-isomorphism $\phi: A\otimes_{C_0(T)} C_0(T,\mathcal{K})\to A$. As above, let $\rho_s'=\zeta^{-1}(\rho_s)\circ\tau\in\mathrm{Out}\big(C_0(T,\mathcal{K})\big)$. Then

$$\begin{aligned} d_A\big(\mathrm{Ad}(\phi)(\gamma\otimes\rho')\big) &= d_{A\otimes_{C_0(T)}C_0(T,\mathcal{K})}\big(\gamma\otimes\rho'\big) \\ &= d_A(\gamma)d_{C_0(T,\mathcal{K})}(\rho') \qquad \text{by Lemma 4.6(2)} \end{aligned}$$

which by definition of d_2' is

$$= d_A(\gamma)d_2'(\rho),$$

which is trivial by construction. The result now follows from Lemma 4.6(1). □

Theorem 4.9. *Suppose that $\ell: G\to\mathrm{Homeo}(T)$ is induced by a second countable locally compact transformation group (G,T) with $H^2(T;\mathbb{Z})$ countable. Then the image of the Forgetful Homomorphism F is exactly those classes $\mathfrak{c}\in H^3(T;\mathbb{Z})$ which lie in $H^3(T;\mathbb{Z})^G$ and which satisfy $d_2(\mathfrak{c})=0=d_3(\mathfrak{c})$. In particular, a stable continuous-trace C^*-algebra A with spectrum T and Dixmier-Douady class $\delta(A)$ admits an action $\alpha: G\to\mathrm{Aut}(A)$ covering ℓ if and only if $\delta(A)\in H^3(T;\mathbb{Z})^G$, $d_2\big(\delta(A)\big)=0$, and $d_3\big(\delta(A)\big)=0$.*

Proof. The first statement follows from the second. Furthermore, the necessity of these conditions is evident. So suppose that $d_3\big(\delta(A)\big)=0$. By Lemma 4.8 there is a twisted action (β,u) on A such that $h(\beta_s)=\ell_s$. Using the stabilization trick [22, Theorem 3.4], we note that $\beta\otimes i$ is exterior equivalent (see [22, Definition 3.1]) to an (ordinary) action α on $A\otimes\mathcal{K}$. Since $A\otimes\mathcal{K}$ is $C_0(T)$-isomorphic to A, we are

done once we show that $h(\alpha_s) = h(\beta_s)$. But, for each fixed s, $\beta_s \otimes i$ differs from α_s by an inner automorphism $\operatorname{Ad} v_s$. Then if $\pi \in \hat{A}$,

$$\begin{aligned} s^{-1} \cdot (\pi \otimes i) &= \pi \otimes i \circ \alpha_s = \pi \otimes i \circ \operatorname{Ad} v_s \circ \beta_s \otimes i \\ &= \pi \otimes i(v_s)\big(\pi \otimes i \circ \beta_s \otimes i\big)\pi \otimes i(v_s^*) \\ &\sim \pi \otimes i \circ \beta_s \otimes i = (s^{-1} \cdot \pi) \otimes i, \end{aligned}$$

which completes the proof. □

5. The Structure Theorem

Theorem 5.1. *Suppose (G, T) is a second countable locally compact transformation group such that $H^2(T; \mathbb{Z})$ is countable. There are homomorphisms*

$$\begin{aligned} &d_2 : H^3(T; \mathbb{Z})^G \to H^2\big(G, H^2(T; \mathbb{Z})\big), \\ &d_2' : H^1\big(G, H^2(T; \mathbb{Z})\big) \to H^3\big(G, C(T, \mathbb{T})\big), \\ &d_2'' : H^2(T; \mathbb{Z})^G \to H^2\big(G, C(T, \mathbb{T})\big), \text{ and} \\ &d_3 : \ker(d_2) \to H^3\big(G, C(T, \mathbb{T})\big) / \operatorname{Im}(d_2'), \end{aligned}$$

with the following properties. (Indeed, d_2, d_2', and d_3 are the homomorphisms defined in the previous section.)

1. *The homomorphism $F : [A, \alpha] \mapsto \delta(A)$ of $\operatorname{Br}_G(T)$ into $H^3(T, \mathbb{Z})^G$ has range $\ker(d_3)$, and kernel consisting of all classes of the form $[C_0(T, \mathcal{K}), \alpha]$.*
2. *Let $\zeta : \operatorname{Aut}_{C_0(T)} C_0(T, \mathcal{K}) \to H^2(T; \mathbb{Z})$ be the homomorphism of* [28, Theorem 2.1]. *Then the homomorphism $\eta : \ker(F) \to H^1\big(G, H^2(T; \mathbb{Z})\big)$ defined by*
$$\eta\big(C_0(T, \mathcal{K}), \alpha\big)(s) = \zeta(\alpha_s \circ \tau_s^{-1}),$$
has range $\ker(d_2')$.
3. *For each cocycle $\omega \in Z^2\big(G, C(T, \mathbb{T})\big)$, choose a Borel map $u : G \to UM\big(C_0(T, \mathcal{K})\big)$ satisfying*
$$u_s \tau_s(u_t) = \omega(s, t) u_{st}, \tag{5.1}$$
and define $\xi(\omega) = [C_0(T, \mathcal{K}), \operatorname{Ad} u \circ \tau]$. Then ξ is a well-defined homomorphism of $H^2\big(G, C(T, \mathbb{T})\big)$ into $\operatorname{Br}_G(T)$ with $\operatorname{Im}(\xi) = \ker(\eta)$ and $\ker(\xi) = \operatorname{Im}(d_2'')$.

Proof. It follows from Theorem 4.9 that the image of F is $\ker(d_3)$. If $(A, \alpha) \in \ker(F)$, then $\delta(A) = 0$ and $A \otimes \mathcal{K}$ is $C_0(T)$-isomorphic to $C_0(T, \mathcal{K})$. Thus

$$(A, \alpha) \sim (A \otimes \mathcal{K}, \alpha \otimes i) \sim \big(C_0(T, \mathcal{K}), \alpha'\big),$$

where $\alpha' = \theta \circ (\alpha \otimes i) \circ \theta^{-1}$ for some $C_0(T)$-isomorphism θ. This proves part (1).

If $[A, \alpha] = [C_0(T, \mathcal{K}), \alpha'] \in \ker(F)$, then we want to define

$$\eta\big([A, \alpha]\big) = \big[s \mapsto \zeta_{C_0(T, \mathcal{K})}(\alpha_s' \circ \tau_s^{-1})\big], \tag{5.2}$$

and, we need to verify that η is well defined. So, suppose that $(A,\alpha)\sim\big(C_0(T,\mathcal{K}),\beta\big)$ as well; α' and β are outer conjugate over T by Lemma 3.1. But if α' is exterior equivalent to β, say $\beta_s=\mathrm{Ad}(u_s)\circ\alpha'_s$, then

$$\zeta\big(\big(\mathrm{Ad}(u_s)\circ\alpha'_s\big)\circ\tau_s^{-1}\big)=\zeta\big(\mathrm{Ad}(u_s)\circ(\alpha'_s\circ\tau_s^{-1})\big)=\zeta(\alpha'_s\circ\tau_s^{-1}).$$

On the other hand, if β is conjugate to α', say $\beta_s=\Phi\circ\alpha'_s\circ\Phi^{-1}$ with $\Phi\in\mathrm{Aut}_{C_0(T)}\big(C_0(T,\mathcal{K})\big)$, then

$$\zeta(\Phi\circ\alpha'_s\circ\Phi^{-1}\circ\tau_s^{-1})=\zeta(\Phi\circ\alpha'_s\circ\tau_s^{-1}\circ\tau_s\circ\Phi^{-1}\circ\tau_s^{-1})$$

which, since the range of ζ is abelian, is

$$\begin{aligned}&=\zeta\big((\alpha'_s\circ\tau_s^{-1})\circ\tau_s\circ\Phi^{-1}\circ\tau_s^{-1}\circ\Phi\big)\\&=\zeta(\alpha'_s\circ\tau_s^{-1})\zeta(\tau_s\circ\Phi^{-1}\circ\tau_s^{-1})\zeta(\Phi)\end{aligned}$$

which, since ζ is equivariant, is

$$=\zeta(\alpha'_s\circ\tau_s^{-1})s\cdot\zeta(\Phi)^{-1}\zeta(\Phi). \tag{5.3}$$

Thus image of α and β differ by a coboundary in $B^1\big(G,H^2(T;\mathbb{Z})\big)$ and (5.2) gives a well-defined map of $\ker(F)$ into $H^1\big(G,H^2(T;\mathbb{Z})\big)$. It is not hard to check, using [29, Proposition 3.10], that η is a homomorphism. Furthermore, $d_2'\big([\rho]\big)=0$ if and only if there is a twisted action (α,u) on $C(X,\mathcal{K})$ such that $\zeta(\alpha_s\circ\tau_s^{-1})=\rho_s$. Thus if $[\rho]=\eta(A,\alpha)$, then certainly $d_2'\big([\rho]\big)=0$. On the other hand, if $d_2'\big([\rho]\big)=0$, then let (α,u) be a twisted action with $\zeta(\alpha\circ\tau^{-1})=\rho$. Then the stabilization trick [22, Theorem 3.4] implies that there is an action β of G on $C_0(T,\mathcal{K})\otimes\mathcal{K}$ which is exterior equivalent to $\alpha\otimes i$: say $\beta_s=\mathrm{Ad}(v_s)\circ(\alpha_s\otimes i)$. Then $\beta_s\circ\tau_s^{-1}=\mathrm{Ad}(v_s)\circ(\alpha_s\otimes i)\circ\tau_s^{-1}=\mathrm{Ad}(v_s)\circ(\alpha_s\circ\tau_s^{-1})\otimes i$, so

$$\zeta(\beta_s\circ\tau_s^{-1})=\zeta(\alpha_s\circ\tau_s^{-1}\otimes i)=\zeta(\alpha_s\circ\tau_s^{-1})=\rho_s.$$

Since $C_0(T,\mathcal{K})\otimes\mathcal{K}$ is $C_0(T)$-isomorphic to $C_0(T,\mathcal{K})$, say by ϕ, we have $\eta\big(C_0(T,\mathcal{K}),\phi\circ\beta\circ\phi^{-1}\big)=[\rho]$. Thus, the image of η is equal to $\ker(d_2')$ as required. This proves (2).

For convenience, let $A=C_0(T,\mathcal{K})$. To define ξ, we first need to note that for every $\omega\in Z^2\big(G,C(T,\mathbb{T})\big)$ there is a Borel map $u:G\to\mathcal{UM}(A)$ satisfying (5.1). However, it was shown in the proof of [15, Proposition 3.1] that

$$\big(u_t^\omega(x)f\big)(s)=\omega(s,t)(s\cdot x)f(st)\qquad\text{for } f\in L^2(G)$$

gives for each $t\in G$ a strongly continuous map $u_t^\omega:T\to\mathcal{U}\big(L^2(G)\big)$, defining a unitary multiplier u_t^ω of $C_0\big(T,\mathcal{K}\big(L^2(G)\big)\big)$, and that $u^\omega:G\to\mathcal{UM}\big(C_0\big(T,\mathcal{K}\big(L^2(G)\big)\big)\big)$ is then a Borel map satisfying (5.1). If G is infinite, then $u=u^\omega$ is the desired map. Otherwise, we can stabilize and let $u=u^\omega\otimes i$. In any case, it is clear that $(A,\mathrm{Ad}(u)\circ\tau)$ is an element of $\mathfrak{Br}_G(T)$. If $v:G\to\mathcal{UM}(A)$ also satisfies (5.1), then $s\mapsto u_sv_s^*$ gives an exterior equivalence between $\mathrm{Ad}(u)\circ\tau$ and $\mathrm{Ad}(v)\circ\tau$, so $[A,\mathrm{Ad}(u)\circ\tau]$ is independent of the choice of u. Since we can absorb a coboundary in $B^2\big(G,C(T,\mathbb{T})\big)$ into the unitary u without changing $\mathrm{Ad}(u)\circ\tau$, we have a well-defined class $\xi\big([\omega]\big)$ in $\mathrm{Br}_G(T)$, and another routine argument shows that ξ is a homomorphism.

To see that the image of ξ is the kernel of η, note that $(\mathrm{Ad}(u)\circ\tau)\circ\tau^{-1}$ consists of inner automorphisms, and hence $\eta\circ\xi\big([\omega]\big)=\eta\big(A,\mathrm{Ad}(u)\circ\tau\big)$ is identically zero. On the other hand, if $\eta(A,\alpha)=0$, then there exists $\phi\in\mathrm{Aut}_{C_0(T)}(A)$ such that $\zeta(\alpha_s\circ\tau_s^{-1})=\zeta(\phi)s\cdot\zeta(\phi)^{-1}=\zeta(\tau_s\circ\phi^{-1}\circ\tau_s^{-1})$ for all $s\in G$. Thus $\phi^{-1}\circ\alpha_s\circ\phi$ differs from τ_s by inner automorphisms $\mathrm{Ad}(u_s)$; we can choose $u:G\to\mathcal{UM}(A)$ to be Borel, u satisfies (5.1) for some cocycle $\omega\in Z^2\big(G,C(T,\mathbb{T})\big)$, and ϕ^{-1} is an isomorphism taking (A,α) to $(A,\mathrm{Ad}(u)\circ\tau)$. Thus, $[A,\alpha]=[A,\mathrm{Ad}(u)\circ\tau]=\xi\big([\omega]\big)$ lies in the image of ξ.

To define the homomorphism d_s'', we choose an automorphism $\phi\in\mathrm{Aut}_{C_0(T)}(A)$ such that $\zeta(\phi)\in H^2(T;\mathbb{Z})^G$, which means precisely that $[\phi]=s\cdot[\phi]:=[\tau_s\circ\phi\circ\tau_s^{-1}]$. Then $s\mapsto\tau_s\circ\phi\circ\tau_s^{-1}\circ\phi^{-1}$ is a Borel map of G into $\mathrm{Inn}(A)$, and there is a Borel map $u:G\to\mathcal{UM}(A)$ such that $\mathrm{Ad}(u_s)\circ\tau_s\circ\phi=\phi\circ\tau_s$. The usual argument shows that $\mathrm{Ad}\big(u_s\tau_s(u_t)\big)=\mathrm{Ad}(u_{st})$, so u satisfies (5.1) for some cocycle $\omega\in Z^2\big(G,C(T,\mathbb{T})\big)$, and ϕ gives an isomorphism of (A,τ) onto the system $\big(A,\mathrm{Ad}(u)\circ\tau\big)$ representing $\xi\big([\omega]\big)$. The choice of u was unique up to multiplication by a function $\rho:G\to C(T,\mathbb{T})$, so ω is unique up to multiplication by $\partial\rho$; choosing a different representative $\mathrm{Ad}(v)\circ\phi$ for $\zeta(\phi)$ would force us to use $v^*u_s\tau_s(v)$ in place of u_s, which would not change ω. Thus we have a well-defined class $[\omega]=d_2''\big(\zeta(\phi)\big)$ in $H^2\big(G,C(T,\mathbb{T})\big)$. If $\mathrm{Ad}(u_s)\circ\tau_s\circ\phi=\phi\circ\tau_s$ and $\mathrm{Ad}(v_s)\circ\tau_s\circ\psi=\psi\circ\tau_s$, then

$$\mathrm{Ad}\big(\phi(v_s)u_s\big)\circ\tau_s\circ(\phi\circ\psi)=(\phi\circ\psi)\circ\tau_s,$$

and a routine calculation using this shows that $d_2''\big(\zeta(\phi\circ\psi)\big)=d_2''\big(\zeta(\phi)\big)+d_2''\big(\zeta(\psi)\big)$.

We have already seen that

$$\xi\Big(d_2''\big(\zeta(\phi)\big)\Big)=[A,\mathrm{Ad}(u)\circ\tau]=[A,\tau],$$

so $\xi\circ d_2''=0$. Conversely, if $\xi\big([\omega]\big)=[A,\mathrm{Ad}(u)\circ\tau]=0$ in $\mathrm{Br}_G(T)$, then Lemma 3.1 gives a $C_0(T)$-automorphism of A such that $\phi^{-1}\circ\big(\mathrm{Ad}(u)\circ\tau\big)\circ\phi$ is exterior equivalent to τ. But if v is a τ-1-cocycle such that

$$\phi^{-1}\circ\big(\mathrm{Ad}(u_s)\circ\tau_s\big)\circ\phi=\mathrm{Ad}(v_s)\circ\tau_s,\tag{5.4}$$

then $\mathrm{Ad}\big(\phi(v_s^*)u_s\big)\circ\tau_s\circ\phi=\phi\circ\tau_s$ and a quick calculation using (5.4) shows that

$$\big(\phi(v_s^*)u_s\big)\tau_s\big(\phi(v_t^*)u_t\big)=\omega(s,t)\phi(v_{st}^*)u_{st},$$

so that $[\omega]=d_2''\big(\zeta(\phi)\big)$. $\square$

6. Examples and Applications

6.1. Actions of $\mathbb{R}$. When $G=\mathbb{R}$, we can sharpen the conclusion of Theorem 4.9 considerably, and we obtain the generalization of [31, Theorem 4.12] mentioned in the introduction.

Corollary 6.1. *Suppose that $(\mathbb{R},T)$ is a second countable locally compact transformation group with $H^1(T;\mathbb{Z})$ and $H^2(T;\mathbb{Z})$ countable, and A is a stable continuous-trace C^*-algebra with spectrum T. Then there is always, up to exterior equivalence, exactly one action $\alpha:\mathbb{R}\to\mathrm{Aut}(A)$ covering the given action on T.*

Proof. Since there is an action on A covering the given action on T if and only if $\delta(A)$ belongs to the range of F, we have to prove that F is surjective. By Theorem 4.9, this is equivalent to proving that $H^3(T;\mathbb{Z})^{\mathbb{R}} = H^3(T;\mathbb{Z})$, that $d_2 = 0$, and that $d_3 = 0$. The connectedness of $\mathbb{R}$ implies that l_s is homotopic to $l_e = \mathrm{id}$, and consequently that $(l_s)^* = \mathrm{id}$ for all s, giving $H^3(T;\mathbb{Z})^{\mathbb{R}} = H^3(T;\mathbb{Z})$. The homomorphism d_2 is 0 because $H^2(\mathbb{R}, M)$ is trivial for any discrete $\mathbb{R}$-module M [44, Theorem 4], and we are assuming that $M = H^2(T,\mathbb{Z})$ is countable. Finally, Theorem 4.1 of [31] says that $H^3(\mathbb{R}, C(T,\mathbb{T})) = 0$, and therefore d_3 is also 0. $\square$

6.2. Free and Proper Actions. We now suppose that G acts freely and properly on T; in other words, that $s\cdot x = x$ if and only if $s = e$, and that $(s,x)\mapsto(s\cdot x, x)$ is a proper map of $G\times T$ into $T\times T$. (If G is a Lie group, it follows from a theorem of Palais [24, Theorem 4.1] that these are precisely the locally trivial principal G-bundles.) Let $p : T \to T/G$ denote the orbit map. By [31, Theorem 1.1], every $(A,\alpha) \in \mathfrak{Br}_G(T)$ in which A is stable is equivalent to an element of the form $(p^*B, p^*\,\mathrm{id}) = \big(C_0(T)\otimes_{C(T/G)} B, \tau\otimes_{C_0(T/G)}\mathrm{id}\big)$—indeed, we can take B to be the crossed product $A\rtimes_\alpha G$. Thus the orbit map p induces a homomorphism $p^* : B \mapsto (p^*B, p^*\mathrm{id})$ of $\mathrm{Br}(T/G)$ onto $\mathrm{Br}_G(T)$. Since the crossed product map $\rtimes_\alpha : (A,\alpha)\mapsto [A\rtimes_\alpha G]$ is well-defined on $\mathrm{Br}_G(T)$ by the Combes–Curto-Muhly-Williams Theorem, and

$$p^*B\rtimes_{p^*\mathrm{id}} G \cong \big(C_0(T)\rtimes_\tau G\big)\otimes_{C(T/G)} B \cong C_0(T/G,\mathcal{K})\otimes_{C(T/G)} B \cong B\otimes\mathcal{K},$$

the map $\rtimes_\alpha$ is an inverse for p^*, and p^* is an isomorphism.

6.3. Trivial Actions. If G acts trivially on T, then $F : \mathrm{Br}_G(T) \to \mathrm{Br}(T)$ is trivially surjective (given A, take (A,id)), and a quick look at the definition in Section 5 shows that the map $d_2'' : H^2(T;\mathbb{Z})^G \to H^2(G, C(T,\mathbb{T}))$ is zero. Thus our structure theorem gives an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^2(G,C(T,\mathbb{T})) \arrow[r, "\xi"] & \ker(F) = \{\, [C_0(T,\mathcal{K}),\alpha] \,\} \arrow[dl, "\eta"'] \\ & \mathrm{Hom}\big(G, H^2(T,\mathbb{Z})\big) \arrow[r, "d_2'"] & H^3(G, C(T,\mathbb{T})). \end{tikzcd}$$

If G is also connected, then every action $\alpha : G\to \mathrm{Aut}_{C_0(T)} C(T,\mathcal{K})$ has range lying in the open subgroup $\mathrm{Inn}\, C(T,\mathcal{K})$, so $\eta = 0$ and $H^2(G, C(T,\mathbb{T}))$ classifies the actions of G on $C(T,\mathcal{K})$ (and, by Remark 3.7, any other stable continuous-trace algebra with spectrum T); see [31, §0]. On the other hand, if $G = \mathbb{Z}$, $H^2(G, C(T,\mathbb{T})) = 0$, and η is an isomorphism of $\ker(F)\cong \mathrm{Aut}_{C_0(T)} C_0(T,\mathcal{K})$ onto $\mathrm{Hom}(G, H^2(T;\mathbb{Z})) = H^2(T;\mathbb{Z})$ by [28, Theorem 2.1]; if $G = \mathbb{R}$, all the groups in the sequence vanish, and $\ker(F)$ is trivial.

There are two extreme special cases. When G is trivial, we recover the Dixmier-Douady isomorphism $\mathrm{Br}(T)\cong H^3(T;\mathbb{Z})$ (see Remark 3.4). When the space T consists of a single point, the elements of $\mathfrak{Br}_G(T)$ are systems $(\mathcal{K}, G, \alpha)$, and we recover from Theorem 5.1 the parameterization of actions of G on $\mathcal{K}$ by the Moore cohomology group $H^2(G,\mathbb{T})$.

6.4. N-proper systems. Suppose G is abelian and there is a closed subgroup N such that $s \cdot x = x$ if and only if $x \in N$, so that G/N acts freely on T. We suppose that $p : T \to T/G$ is a locally trivial principal G/N-bundle. If N is a compactly generated group with $H^2(N, \mathbb{T}) = 0$, and $(A, \alpha) \in \mathfrak{Br}_G(T)$, then $\alpha|_N$ is locally unitary [40, Corollary 2.2], so the system (A, α) is N-principal in the sense of [34, 37, 35]. Provided the quotient map $\widehat{G} \to \widehat{N}$ has local cross-sections (equivalently, $\widehat{G}$ is locally trivial as an $N^{\perp}$-bundle), then Proposition 4.8, Corollary 4.4 and Theorem 6.3 of [35] imply that the Dixmier-Douady invariant of [35] induces an isomorphism of $\mathrm{Br}_G(T)$ onto the equivariant sheaf cohomology group $H^2_G(T, \mathcal{S})$ studied in [36]. The Gysin sequence of [36] then implies that we have an exact sequence

$$\to \mathrm{Br}(T/G) \xrightarrow{p^*} \mathrm{Br}_G(T) \xrightarrow{b} H^1(T/G, \widehat{N}) \longrightarrow H^3(T/G, \mathcal{S}) \to \cdot$$

The homomorphism p^* takes a continuous-trace algebra B with spectrum T/G to $(p^*B, p^* \mathrm{id})$, and the homomorphism b takes (A, α) to the class of the principal $\widehat{N}$-bundle $(A \rtimes_\alpha G)^\wedge \to T/G$. Thus taking $N = \{e\}$ gives a special case of Section 6.2 concerning free actions, and taking $G = N$ gives a short exact sequence

$$0 \longrightarrow \mathrm{Br}(T) \longrightarrow \mathrm{Br}_G(T) \xrightarrow{\zeta} H^1(T, \widehat{G}) \longrightarrow 0$$

summarizing the results of [29] for the case where G acts trivially on T. Various other special cases are considered in the last section of [36]. However, we stress that the group $H^2_G(T, \mathcal{S})$ is *not* a true equivariant cohomology group in the sense of [13]: if $H^2(N, \mathbb{T}) \neq 0$, then the systems (A, α) which are locally Morita equivalent to $\big(C_0(T), \tau\big)$, and hence classifiable by their Dixmier-Douady class in $H^2_G(T, \mathcal{S})$, form a proper subgroup of $\mathrm{Br}_G(T)$.

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